ON A RECURSIVE FORMULA FOR BINOMIAL COEFFICIENTS

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ABSTRACT. We demonstrate that the binomial coefficients $B(n,k)$ satisfy the recursive formula

$$B(n,k) = B(n-1,k-1) + B(n-1,k).$$

Moreover, we prove that together with the initial conditions $B(n,0) = B(n,n) = 1$ our formula can be used to calculate all binomial coefficients.

1. INTRODUCTION

The binomial coefficients $B(n,k)$ arise in the expansions of the polynomial $(a+b)^n$. In fact, $B(n,k)$ is the coefficient of the term $a^k b^{n-k}$ is the said expansion. The case $n = 2$ dates back to Euclid who expressed it in geometric terms [1]. The general formula

$$B(n,k) = \frac{n!}{k!(n-k)!} \qquad n \geq 0,\ 0 \leq k \leq n, \tag{1}$$

is attributed to Newton who reportedly published it on his gravestone, although this claim is disputed by some authors [2].

Formula (1), although explicit, is not very convenient in abacus computations, as one has to multiply and divide large numbers. In the present manuscript we prove that $B(n,k)$ satisfy certain recurrence relation which, together with few initial values, reduce the task of computing the binomial coefficients to a series of additions. Our main theorem can be formulated as follows

Theorem 1. *The binomial coefficients $B(n,k)$ defined by formula (1) satisfy*

$$B(n,k) = B(n-1,k-1) + B(n-1,k), \qquad 1 \leq k \leq n-1. \tag{2}$$

Together with the initial conditions $B(n,0) = B(n,n) = 1$ recursion (2) completely specifies the binomial coefficients.

2. Proof of the main result

Proof. Using equation (1) we obtain

$$B(n-1,k-1) = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k-1)!}\frac{1}{n-k},$$
$$B(n-1,k) = \frac{(n-1)!}{k!(n-k-1)!} = \frac{(n-1)!}{(k-1)!(n-k-1)!}\frac{1}{k}.$$

Adding the two together we calculate

$$\begin{aligned} B(n-1,k-1)+B(n-1,k) &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\frac{n}{k(n-k)} \\ &= \frac{n!}{k!(n-k)!} \\ &= B(n,k). \end{aligned}$$

This proves recursion (2). We notice that the right-hand side of (2) only makes sense when $1 \le k \le n-1$, since the original binomial $B(n,k)$ is only defined when $0 \le k \le n$.

To demonstrate that the above recursion fully specifies the binomial coefficients when supplemented with the initial conditions $B(n,0) = B(n,n) = 1$ we employ induction on n. For $n = 0$ and $n = 1$ all the binomial coefficients are specified by the initial data:

$$B(0,0) = 1, \qquad B(1,0) = 1, \qquad B(1,1) = 1.$$

Assume all values for $n = N-1$ have been found. Then we can find values of $B(N,k)$ for $k = 1,\ldots,N-1$ from the recursion. For the remaining values of k, $k = 0$ and $k = N$, the values are again given by the initial data:

$$B(N,0) = 1, \qquad B(N,N) = 1.$$

This completes the proof of the theorem. □

3. Conclusions and Outlook

We have shown that the binomial coefficients satisfy a recurrence relation which can be used to speed up abacus calculations. Our approach raises an important question: what can be said about the solution of the recurrence (2) if the initial data is different? For example, if $B(n,0) = 1$ and $B(n,n) = -1$, do coefficients $B(n,k)$ stay bounded for all n and k?

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Useful discussions with B. Pascal regarding arrangements of $B(n, k)$ in a hexagonal pattern are gratefully acknowledged.

References

[1] Euclid, *Elements*, 34th ed., Macedonian Press, Thebes, 1976.
[2] F. Cajori, *Was the binomial theorem engraven on Newton's monument?*, Bull. Amer. Math. Soc. **1** (1894), 52–54 .

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