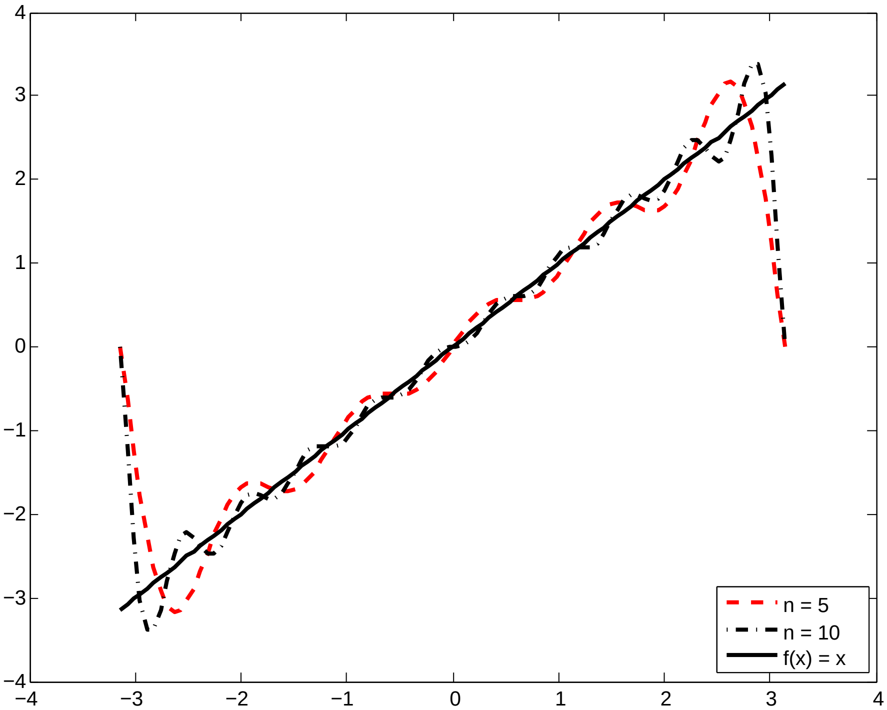
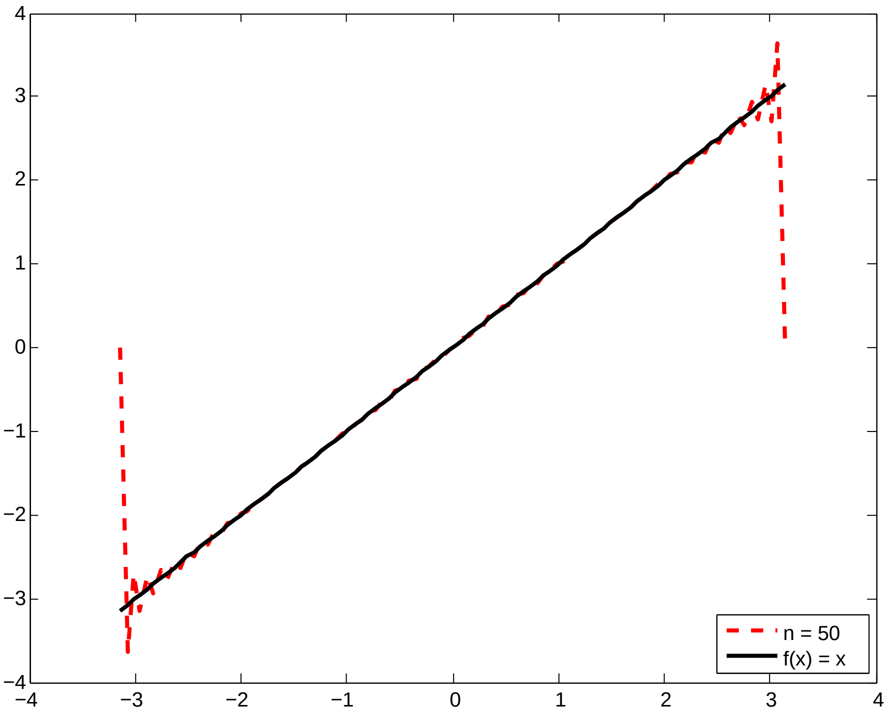
Fourier Series

Here is the graph of $f(x) = x$ together with the $n-$th partial sums for $n = 5, 10$ over $[-\pi, \pi]$:

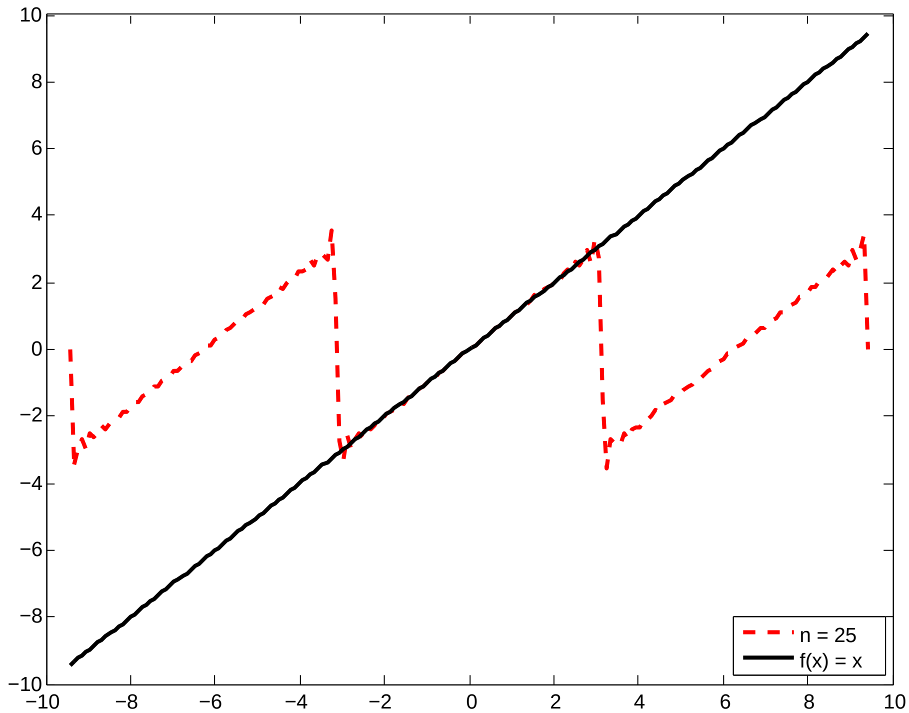


Next, here is the graph of f and the $n = 50$ partial sum:



See there is trouble at the ends $x = \pm\pi$.

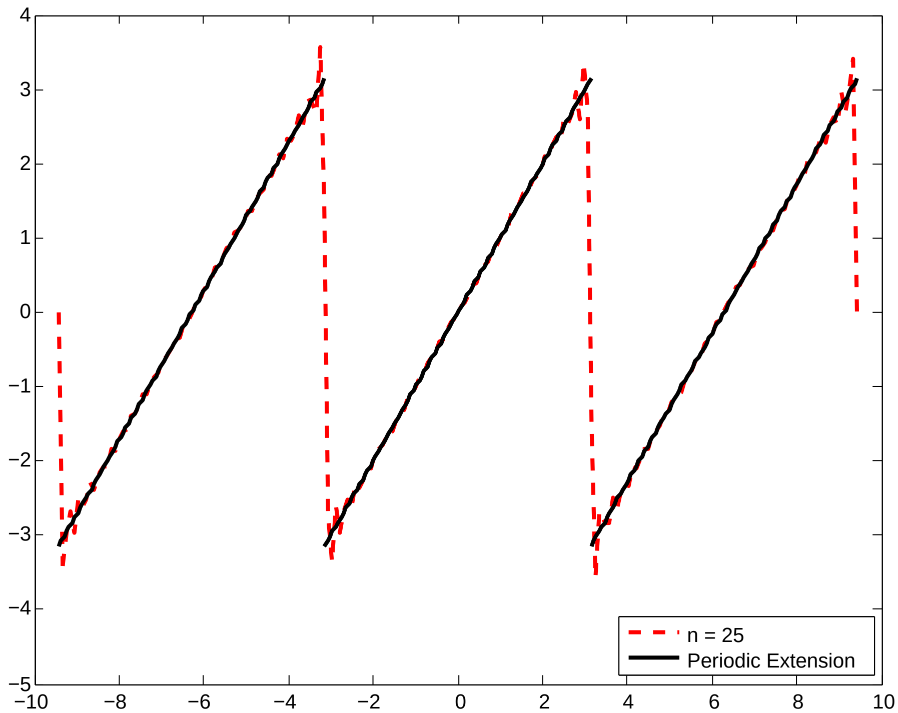
The Fourier series does *not* represent $f(x) = x$ outside $[-\pi, \pi]$:



Reason:

- The Fourier series is $2\pi-$periodic.
- The function $f(x) = x$ is not $2\pi-$periodic.
- In fact, the Fourier series represents the $2\pi-$periodic extension of $f(x) = x, \quad x \in [-\pi, \pi]$.
- Recall the graph of the $2\pi-$periodic extension is repeated carbon copies to the left and right of the graph of $f(x) = x, \quad x \in [-\pi, \pi]$.

Here is the periodic extension of $f(x) = x$, $[-\pi, \pi]$ and partial sum $n = 50$:



- Observe there are overshoots of the Fourier series at the points $x = \pm\pi$, $\pm 3\pi$. This always happens at the discontinuities of the periodic extension of a function. This is known as the *Gibbs Phenomenon*.

- Notice the Fourier series converges to the "half-way" value at the points $x = \pm\pi$, $\pm 3\pi$. This usually happens; in fact the following holds.

General Principle: Let $f(x)$ be $2\pi-$periodic.

- f continuous at $a \implies f(a) =$ Fourier series at a

- f not continuous at $a \implies$ Fourier series at a converges to the average of the left and right limits of f at a:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(na) + b_n \sin(na) = \frac{1}{2}\left[\lim_{x \to a^-} f(x) + \lim_{x \to a^a} f(x)\right]$$