THE LIDSKII TRACE PROPERTY AND THE NEST APPROXIMATION PROPERTY IN BANACH SPACES

T. FIGIEL AND W. B. JOHNSON

Abstract. For a Banach space $X$, the Lidskii trace property is equivalent to the nest approximation property; that is, for every nuclear operator on $X$ that has summable eigenvalues, the trace of the operator is equal to the sum of the eigenvalues if and only if for every nest $\mathcal{N}$ of closed subspaces of $X$, there is a net of finite rank operators on $X$, each of which leaves invariant all subspaces in $\mathcal{N}$, that converges uniformly to the identity on compact subsets of $X$. The Volterra nest in $L_p(0,1)$, $1 \leq p < \infty$, is shown to have the Lidskii trace property. A simpler duality argument gives an easy proof of the result [ALL, Theorem 3.1] that an atomic Boolean subspace lattice that has only two atoms must have the strong rank one density property.

1. Introduction

This paper originates with two interesting structural theorems about Hilbert spaces. In 1959 Lidskii [L] proved that every nuclear (i.e., trace class) operator on a (complex) Hilbert space has its trace equal to the sum of its eigenvalues. (Pisier [P1] pointed out that Grothendieck was aware of this result somewhat earlier [G2].) In 1966 J. Erdos [E1] proved that if $\mathcal{N}$ is a nest of closed subspaces of a Hilbert space $H$, then the identity operator on $H$ is the strong limit of a net of finite rank contractions each of which leaves invariant every subspace in $\mathcal{N}$. It is interesting to investigate which Banach spaces have these desirable properties. For the trace property, some adjustment must be made, because in [JKMR] it is proved that on every Banach space that is not isomorphic to a Hilbert space, there is a nuclear operator whose eigenvalues are not summable. Therefore, following [JS2], we say that a complex Banach space $X$ has the Lidskii trace property or (L) provided that whenever $T$ is a nuclear operator on $X$ whose
eigenvalues are (absolutely) summable, the trace of $T$ is equal to the sum of its eigenvalues (of course, repeated according to multiplicity).

It is known that if $X$ has the Lidskii trace property, then $X$ has the hereditary approximation property (HAP); that is, every subspace of $X$ has the approximation property (see, e.g., [JS1]). Consequently, as discussed in [JS2], such an $X$ must in some sense be very close to a Hilbert space; in particular, it must have, for every $\epsilon > 0$, cotype $2 + \epsilon$ and type $2 - \epsilon$.

The very strong approximation property of Hilbert spaces discovered by Erdos [E1] suggests the following definitions. Given a nest $\mathcal{N}$ of (closed) subspaces of a Banach space $X$, say that $X$ has the $\mathcal{N}$-approximation property if there is a net of finite rank operators on $X$ that converges uniformly to the identity on compact sets and leave invariant all subspaces in $\mathcal{N}$. If these finite rank operators can be chosen all to have norm at most $\lambda$, we say that $X$ has the $\lambda$-$\mathcal{N}$-approximation property. If $X$ has the $\mathcal{N}$-approximation property (respectively, the $\lambda$-$\mathcal{N}$-approximation property) for all nests, we say that $X$ has the nest approximation property (respectively, nest $\lambda$-approximation property or $\lambda$-NAP). If $X$ has the $\lambda$-NAP for some $\lambda$, then $X$ is said to have the bounded NAP, and $X$ is said to have the metric NAP if it has the 1-NAP. This terminology is consistent with more familiar types of approximation properties. Fortunately, if a reflexive space has the $\mathcal{N}$-AP, then it has the 1-$\mathcal{N}$-AP. The roots of this go back to Grothendieck [G1] (see also [LT, Theorem 1.e.15]), who proved that a reflexive space that has the approximation property must have the metric approximation property. The nest version follows e.g. from the work of Godefroy and Saphar [GS, Theorem 1.5] (or see [LO, Corollary 5.3]).

Erdos discovered a connection between his theorem and the Lidskii trace formula. Specifically, he showed [E2] that the trace formula in Hilbert spaces could be proved in a simple way using his nest approximation theorem. There is indeed a close connection: Theorem 3.2 says that a Banach space satisfies (L) if and only if it has the NAP. This would be of only academic interest if only Hilbert spaces had property (L). However, Pisier [P1] introduced a class of Banach spaces called weak Hilbert spaces that satisfy (L), and in [JS2] Pisier’s result was extended to a wider class of spaces that includes some classical spaces. All of these spaces are reflexive and thus all have the metric NAP.

Most of the proof of the equivalence of the NAP to property (L) is valid nest by nest. Given a nest $\mathcal{N}$ of subspaces of $X$, we prove in Theorem 2.1 that $X$ has the $\mathcal{N}$-AP if and only if every quasi-nilponent (which for compact operators just means no non-zero eigenvalues) nuclear operator on $X$ that leaves all subspaces of $\mathcal{N}$ invariant has zero
trace. We use Ringrose’s [R1] structure theory for compact operators to derive Theorem 3.2 from Theorem 2.1.

It is worth noting that prior to our work it was not known how to derive Erdos’ nest approximation theorem from the Lidskii trace formula. It is both amusing and puzzling that we do not see how to check directly that any space satisfying (L) (other than Hilbert spaces, of course) has the NAP–Erdos’ argument for Hilbert spaces uses tools that are simply not available in the Banach space setting. However, for special nests it is possible to check the nest approximation property and to deduce the equivalent trace property from it. In Theorem 4.2 we do this for the Volterra nest in \( L^p, 1 \leq p < \infty \).

Although the main motivation for this paper comes from theorems about Hilbert spaces, we stumbled onto the results presented here because of our work [FJP], [FJ] on the concept of the approximation property for a Banach space \( X \) and a subspace \( Y \). The pair \( (X, Y) \) has the AP provided \( X \) has the \( N \)-AP for the simple nest \( \{Y\} \). Theorem 2.1 was proved for nests with only one element in [FJ], and the proof given there generalizes to give a proof of Theorem 2.1.

Our notation is standard (see e.g. [LT]) except that for convenience we use \( \subset \) for proper subset and \( \subseteq \) when the containment is not necessarily proper. If \( \mathcal{L} \) is a collection of closed subspaces of a Banach space \( X \), \( \text{Alg}(\mathcal{L}) \) denotes the collection of all operators on \( X \) that leave each subspace in \( \mathcal{L} \) invariant. We work with complex Banach spaces because we are concerned with the spectral properties of operators, but the proofs for the non spectral results work in the real case.

2. The equivalence of the NAP with property (L)

Throughout the rest of this paper \( \mathcal{N} \) will denote a complete nest of closed subspaces of a Banach space \( X \) that contains \( \{0\} \) and \( X \). In this context “complete” just means that if \( \mathcal{N}_1 \) is a non-empty subcollection of \( \mathcal{N} \), then \( \cap \mathcal{N}_1 \) and the closure of \( \cup \mathcal{N}_1 \) are both in \( \mathcal{N} \). Since we are interested in operators on \( X \) that leave invariant every element of \( \mathcal{N} \), there is no loss of generality in considering just these nests. For \( Y \in \mathcal{N} \), \( \mathcal{N}_{Y^-} \) is the set of \( Z \in \mathcal{N} \) such that \( Z \) is a proper subspace of \( Y \) and \( Y^- \) denotes \( \cup \mathcal{N}_{Y^-} \). Since \( \mathcal{N} \) is complete, \( Y^- \) is in \( \mathcal{N} \). Clearly \( Y^- \) is either equal to \( Y \) or is the immediate predecessor of \( Y \) in \( \mathcal{N} \). \( \mathcal{F}(X) \) denotes the finite rank operators on \( X \) and \( \mathcal{F}_{\mathcal{N}}(X) \) are the operators in \( \mathcal{F}(X) \) that leave every subspace in \( \mathcal{N} \) invariant.

Our first lemma is a version of Lemma [FJ, Lemma 1] for nests. Part (1) is due to Ringrose [R2] and part (2) is due to Spanoudakis
[Sp, Theorem 2]. Our proof is much simpler than the proof in [Sp] but close in spirit to the proof of [LiLu, Theorem 3.1].

Lemma 1. Let $\mathcal{N}$ be a complete nest of closed subspaces of the Banach space $X$ (with $\{0\}$, $X$ in $\mathcal{N}$, as always), and let $x^*$ and $x$ be non zero elements of $X^*$ and $X$, respectively.
1. $x^* \otimes x \in \mathcal{F}_N(X)$ if and only if there is $Y \in \mathcal{N}$ such that $x \in Y$ and $x^* \in \mathcal{N}^\bot_Y$.
2. If $T \in \mathcal{F}_N(X)$ and $n > 0$ is the rank of $T$, then $T$ is the sum of $n$ rank one elements in $\mathcal{F}_N(X)$.

Proof. To prove (1), let $Y$ be the intersection of all $Z \in \mathcal{N}$ for which $x \in Z$. Then $Y$ is the smallest subspace in $\mathcal{N}$ that contains $x$, which is to say that $x \in Y$ but $x \not\in \cup \mathcal{N}^\bot_Y$, and hence $x^* \in \mathcal{N}_Y^\bot$ if $x^* \otimes x \in \mathcal{F}_N(X)$. This gives the only if part of (1) since operators in $\mathcal{F}_N(X)$ leave $\cup \mathcal{N}^\bot_Y$. The reverse direction is clear.

The proof of (2) is by induction on $n$, the rank of $T$. The case $n = 1$ is obvious. Suppose that (2) is true for some $n = m - 1$ and that $T \in \text{Alg}(\mathcal{N})$ has rank $m$. Let $Y_1$ be the smallest subspace, $Y$, in $\mathcal{N}$ such that $Y \cap TX \neq \{0\}$. ($Y_1$ is just the intersection of all $Y$ in $\mathcal{N}$ for which $Y$ contains a point in the unit sphere of $TX$; this intersection contains a unit vector of $TX$ by compactness.) Let $y_1$ be a non zero vector in $Y_1 \cap TX$ and extend to a basis $y_1, \ldots, y_m$ for $TX$. So you can write $T = \sum_{k=1}^m y_k^* \otimes y_k$. We claim that $y_1^*$ is in $\cup \mathcal{N}_{Y_1}^\bot$. Assuming the claim, we note that by (1) we have that $y_1^* \otimes y_1$ is in $\mathcal{F}_N(X)$, whence so is $T - y_1^* \otimes y_1 = \sum_{k=2}^m y_k^* \otimes y_k$. Thus (2) follows by induction.

To prove the claim that $y_1^*$ is in $\cup \mathcal{N}_{Y_1}^\bot$, first note that $T(\cup \mathcal{N}_{Y_1}^\bot) \subseteq \cup \mathcal{N}_{Y_1}^\bot$ because $T \in \text{Alg}(\mathcal{N})$, and $TX \cap (\cup \mathcal{N}_{Y_1}^\bot) = \{0\}$ by the definition of $Y_1$, so $T(\cup \mathcal{N}_{Y_1}^\bot) = \{0\}$. On the other hand, if the claim were false, there would be $x \in \cup \mathcal{N}_{Y_1}^\bot$ with $\langle y_1^*, x \rangle \neq 0$, which would imply that $T x \neq 0$ by the linear independence of $y_1, \ldots, y_m$.

Theorem 2.1, which gives the dual version of the statement that $(X, \mathcal{N})$ has the AP, is the main result of this paper. $N(X, Z)$ denotes the nuclear operators from $X$ to $Z$ and is abbreviated as $N(X)$ when $X = Z$. In the hypothesis we assume that the space $X$ has the AP in order to formulate the theorem with nuclear operators $N(X)$ rather than with the projective tensor product of $X^*$ with $X$. Given $T \in N(X)$, $\text{tr}(T)$ is the trace of $T$, which is well-defined when $X$ has the AP by Grothendieck’s fundamental result [G1], [LT, Theorem 1.e.4]. We denote by $\tau$ the topology on $L(X)$ of uniform convergence on compact subsets of $X$. 
Theorem 2.1. Let \( \mathcal{N} \) be a complete nest of closed subspaces of the Banach space \( X \) and assume that \( X \) has the AP. The following are equivalent.

1. The pair \((X, \mathcal{N})\) has the AP.
2. For all \( T \in N(X) \) for which \( TY \subseteq Y^- \) for every \( 0 \neq Y \in \mathcal{N} \), we have \( \text{tr}(T) = 0 \).

Proof. (1) \( \Rightarrow \) (2). Assume (2) is false and get \( T \in N(X) \) so that \( TY \subseteq Y^- \) for every \( 0 \neq Y \in \mathcal{N} \) but \( \text{tr}(T) = 1 \). So \( T \in L(X, \tau)^* \) and \( \langle I, T \rangle = 1 \). Let \( F \in \mathcal{F}_\mathcal{N}(X) \). We want to show that \( \langle F, T \rangle = 0 \), which would contradict (1). By Lemma 1, it is enough to check that \( \langle x^* \otimes x, T \rangle = 0 \) when there is \( Y \in \mathcal{N} \) so that \( x^* \in \mathcal{N}^\perp_Y \) and \( x \in Y \). But then \( \langle x^* \otimes x, T \rangle = \langle x^*, Tx \rangle \), so this is clear from the facts that \( x \in Y \) and \( TY \subseteq Y^- \).

(2) \( \Rightarrow \) (1). If (1) is false and \( X \) has the AP, we can separate \( I \) from \( \mathcal{F}_\mathcal{N}(X) \) with a \( \tau \) continuous linear functional on \( L(X) \), which, since \( X \) has the AP, is represented by a nuclear operator \( T \) on \( X \) (see [G1], [LT, Theorems 1.e.3, 1.e.4]).

Then \( \text{tr}(T) = \langle I, T \rangle \neq 0 \) but \( \langle F, T \rangle = 0 \) for all \( F \in \mathcal{F}_\mathcal{N}(X) \). In particular, \( \langle x^* \otimes x, T \rangle = \langle x^*, Tx \rangle = 0 \) if \( Y \in \mathcal{N} \), \( x^* \in \mathcal{N}^\perp_Y \), and \( x \in Y \). For \( Y \in \mathcal{N} \), this holding for all such \( x^* \) and \( x \) just says that \( TY \subseteq \sqcup \mathcal{N}^-_Y = Y^- \), which contradicts (2).

Remark 2.2. In [Sp, Theorem 3] it is shown that \( \mathcal{F}_\mathcal{N}(X) \) is strongly dense in \( \mathcal{A}lg(\mathcal{N}) \) when \( \mathcal{N} \) is any complete nest of subspaces of \( X \). Spanoudakis [Sp, p. 722] then mentions the problem whether \((X, \mathcal{N})\) must have the 1-BAP whenever \( \mathcal{N} \) is a complete nest and \( X \) has the 1-BAP. The answer is of course "no", because then every subspace in \( \mathcal{N} \) would have the 1-BAP. In fact, as mentioned in [FJP], \((X,Y)\) can fail to have the AP when \( X \) is reflexive and both \( X \) and \( Y \) have the AP, because \( X/Y \) might fail the AP (as mentioned earlier, reflexivity implies that any AP condition is equivalent to the corresponding 1-BAP condition). Here, by the main result in [Sz], \( X \) can be \( \ell_p \), \( 1 < p < 2 \).

If we specialize Theorem 2.1 to finite nests, we get the following:

Corollary 1. Let \( \mathcal{N} = \{0 = Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_n = X\} \) be an \( n+1 \)-element nest of closed subspaces of a Banach space \( X \) that has the AP. Then \((X, \mathcal{N})\) has the AP if and only if every \( T \in N(X) \) for which \( TY_{k+1} \subseteq Y_k \) for \( 0 \leq k < n \) has zero trace.

The final result in this section is an immediate consequence of Corollary 1. Item (2) in Corollary 2 says that the spectral trace and the trace agree on the nilpotent nuclear operators on \( X \) if and only if \((X, \mathcal{N})\) has
the AP for every finite nest \( \mathcal{N} \). In Theorem 3.2 we give a version of this for maximal nests.

**Corollary 2.** Let \( X \) be a Banach space that has the AP and \( n \) a non negative integer.
1. \((X, \mathcal{N})\) has the AP for every nest of closed subspaces of \( X \) (containing \( \{0\} \) and \( X \)) that has at most \( n \) elements if and only if whenever \( T \in \mathcal{N}(X) \) and \( T^{n+1} = 0 \), \( T \) must have zero trace.
2. \((X, \mathcal{N})\) has the AP for every finite nest of closed subspaces of \( X \) if and only if whenever \( T \in \mathcal{N}(X) \) is nilpotent, \( T \) must have zero trace.

### 3. Applications: Erdos meets Lidskii

In this section we concentrate on the AP for \((X, \mathcal{N})\) when \( \mathcal{N} \) is a maximal nest of closed subspaces. Of course, maximality implies completeness and also that \( \mathcal{N} \) contains \( X \) and the zero subspace, so Theorem 2.1 applies.

A key ingredient for connecting the nest approximation property to the Lidskii trace property is Ringrose’s [R1] structure theory for compact operators on a Banach space. From [R1, Theorem 2] we have:

**Proposition 1.** Let \( \mathcal{N} \) be a maximal nest of closed subspaces of the Banach space \( X \) and let \( T \) be a compact operator in \( \text{Alg}(\mathcal{N}) \). The following are equivalent.
1. For every \( Y \in \mathcal{N} \), \( TY \subseteq Y^- \).
2. \( T \) is quasi-nilpotent.

In view of Proposition 1, we can restate Theorem 2.1 for maximal nests as Corollary 3.

**Corollary 3.** Let \( \mathcal{N} \) be a maximal nest of closed subspaces of the Banach space \( X \) and assume that \( X \) has the AP. The following are equivalent.
1. The pair \((X, \mathcal{N})\) has the AP.
2. For every quasi-nilpotent \( T \in \mathcal{N}(X) \cap \text{Alg}(\mathcal{N}) \), we have \( \text{tr} (T) = 0 \).

Condition (2) in Corollary 3 says that the trace of a quasi-nilpotent nuclear operator in \( \text{Alg}(\mathcal{N}) \) is equal to its spectral trace (the sum of its eigenvalues). Again using Ringrose’s structure theory for compact operators, we can soup this up to include nuclear operators in \( \text{Alg}(\mathcal{N}) \) that have summable eigenvalues.

**Proposition 2.** Let \( \mathcal{N} \) be a maximal nest of closed subspaces of the Banach space \( X \) and assume that \( X \) has the AP. The following are equivalent.
1. The pair \((X, \mathcal{N})\) has the AP.
2. For every quasi-nilpotent \(T \in N(X) \cap \text{Alg}(\mathcal{N})\), we have \(\text{tr}(T) = 0\).
3. For every \(T \in N(X) \cap \text{Alg}(\mathcal{N})\) whose eigenvalues \(\{\lambda_k\}\) are absolutely summable, we have \(\text{tr}(T) = \sum_k \lambda_k\).

Proof. In view of Corollary 3, we only need to prove \((2) \implies (3)\). The main tool is Ringrose’s \([R1]\) structure theory for compact operators. If \(T\) is a compact operator in \(\text{Alg}(\mathcal{N})\) and \(Y \in \mathcal{N}\), then either \(Y^{-} = Y\) or \(Y^{-}\) has codimension one in \(Y\). In the latter case, there is an eigenvalue \(\lambda_Y\) of \(T\) so that for every \(x \in Y \sim Y^{-}\) we have \(Tx = \lambda_Y x + y_x\) with \(y_x \in Y^{-}\). Let \(\mathcal{N}' := \{Y \in \mathcal{N} : \dim Y/Y^{-} = 1\}\). The collection \(\{\lambda_Y : Y \in \mathcal{N}'\}\) exhausts the eigenvalues of \(T\) repeated according to multiplicity, and so \(\mathcal{N}'\) is countable.

Suppose now that \(\sum_{Y \in \mathcal{N}'} |\lambda_Y| < \infty\). For \(Y \in \mathcal{N}'\) pick \(x_Y \in Y\) of norm one so that the distance of \(x_Y\) to \(Y^{-}\) is close to one. Choose a functional \(x_Y^* \in (Y^{-})^\perp\) with norm close to one so that \(x_Y^*(x_Y) = 1\). Then the linear operator \(S := \sum_{Y \in \mathcal{N}'} \lambda_n x_Y^* \otimes x_Y\) is nuclear since \(\sum |\lambda_k| < \infty\), and, by Lemma 1, is in \(\text{Alg}(\mathcal{N})\). Consequently, \(\mathcal{N}\) is a (necessarily maximal) nest of invariant subspaces for the compact operator \(T - S\). By construction, for every \(Y \in \mathcal{N}'\) we have that \((T - S)Y \subseteq Y^{-}\), which is to say that \(T - S\) is quasi-nilpotent by Proposition 1, and of course nuclear if \(T\) is nuclear.

\[\square\]

Remark 3.1. At the insistence of the editor for [JS2], the proof of the implication \((2) \implies (3)\) in Proposition 2 was included in [JS2] with the permission of the first author (and despite objections by the second author).

Finally we state the “Erdos meets Lidskii” theorem:

**Theorem 3.2.** The following are equivalent for a Banach space \(X\) that has the AP.
1. \(X\) has the nest approximation property.
2. Every quasi-nilpotent nuclear operator on \(X\) has zero trace.
3. If \(T\) is a nuclear operator on \(X\) that has summable eigenvalues, then the trace of \(T\) is equal to the sum of its eigenvalues.
4. Every quasi-nilpotent nuclear operator on \(X\) is the limit in the nuclear norm of finite rank nilpotent operators.

Proof. In view of Ringrose’s comment [R1] that a maximal nest of invariant subspaces of a compact operator on a Banach space \(X\) is a maximal nest of subspaces of \(X\), the equivalence of the first three items in Theorem 3.2 is an immediate consequence of Proposition 2.
The implication $(4) \implies (2)$ is evident from the continuity of the trace on $N(X)$ when $N(X)$ is given the nuclear norm. This leaves:

$(1) \implies (4)$. Let $T$ be a quasi-nilpotent nuclear operator on $X$ and let $N$ be a maximal nest of closed invariant subspaces for $T$. As was remarked earlier, $N$ is then a maximal nest of closed subspaces of $X$ [R1, Theorem 2]. By $(1)$, there is a net $S_\alpha$ in $F_N(X)$ that converges to the identity operator uniformly on compact sets. Proposition 1 implies that each $S_\alpha T$ is quasi-nilpotent, hence nilpotent because they have finite rank, and of course $S_\alpha T$ converges to $T$ in the nuclear norm. ■

Remark 3.3. That condition $(4)$ in Theorem 3.2 is true in a Hilbert space is due to Erdos [E2]. The proof we gave is not essentially different. Condition $(4)$ for Hilbert space is also the third exercise in [D, Chapter 3]. We thank P. Skoufranis for this reference and its solution [S].

4. Examples

Example 4.1. The most obvious examples of maximal nests that satisfy the conditions in Proposition 2 are given by Schauder bases. Given a Schauder basis $\{e_n\}_{n=1}^\infty$ for a Banach space $X$, let $N$ consist of $\{0\}, X$ and span $\{e_1, \ldots, e_n\}_{n=1}^\infty$. Then $N$ is a maximal nest and $X$ has the $\lambda$-$N$ AP, where $\lambda$ is the basis constant of $\{e_n\}_{n=1}^\infty$.

Fix $1 \leq p < \infty$. The Volterra nest $V$ for $L_p := L_p[0,1]$ consists of $\{L_p[0,t] : 0 \leq t \leq 1\}$; Here $L_p[0,t]$ is identified with the $L_p$ functions that vanish off $[0,t]$. The Volterra nest is a maximal continuous nest (a nest $N$ is called continuous if $Y^- = Y$ for every $Y \in N$; all continuous nests containing the zero subspace and the whole space are maximal).

Theorem 4.2. Let $V$ be the Volterra nest for $L_p$, $1 \leq p < \infty$. Then $L_p$ has the $1-V$-AP.

Proof. The case $p = 2$ follows from Erdos’ more general theorem [E1], but the proofs we have seen use Hilbert space techniques that are not obviously adaptable to the case of the Volterra nest in $L_p$, $p \neq 2$. Anyway, the proof we give is very simple.

Since the step functions are dense in $L_p$, it is enough to show that if $\{(a_k, b_k)\}_{k=1}^n$ are disjoint open subintervals of $[0,1]$ and $\varepsilon > 0$, then there is a norm one operator $T \in F_V(L_p)$ so that for each $1 \leq k \leq n$ we have

\[
\|T1_{(a_k,b_k)} - 1_{(a_k,b_k)}\|_p < \varepsilon.
\]

It follows easily that it is enough to get this approximation for each interval $(a_k, b_k)$ as long as the operator (call it $T_k$) is zero on the $L_p$ functions that vanish on $(a_k, b_k)$, for then $T := \sum_{k=1}^n T_k$ will satisfy (1)
and will have norm one. Given one interval \((a, b)\) and \(N \in \mathbb{N}\), consider the subintervals \(I_k := ((a+k(b-a))/N, a+(k+1)(b-a)/N)\), \(0 \leq k < N\). Define \(T\) on \(L_p\) by

\[
Tg = N/(b-a) \sum_{k=0}^{N-2} \int_{I_{k+1}} g(s) \, ds \cdot 1_{I_k}.
\]

Then \(T\) has norm one, \(T\) is zero on \(L_p\) functions that vanishes on \((a, b)\), \(T1_{(a,b)} = 1_{(a,b)-(b-a)/N}\), and each of the summands that define \(T\) are in \(F_v(L_p)\).

Remark 4.3. In [ALWW] the authors discuss the Volterra nest in \(L_p\) as well as more general nests that they term “modeled on subspaces” (or MOS) nests. These are complete nests in which every element is of the form \(1_E L_p\) for some measurable set \(E\). It looks likely that a modification of the argument for Theorem 4.2 will show that \(L_p\) has the 1-\(N\)-AP for every MOS nest \(N\).

5. The strong rank one density property

If \(X\) is a Banach space, a \textit{subspace lattice} on \(X\) is a complete sublattice containing \(\{0\}\) and \(X\) of the lattice of all closed subspaces of \(X\). Here the order is inclusion, so the joins are closed linear spans and meets are intersections. Such objects arise naturally in both operator theory and the theory of bases; see [ALL] for why. A subspace lattice \(\mathcal{L}\) on \(X\) is said to have the \textit{strong rank one density property} provided the algebra generated by the rank one operators in \(\text{Alg} \mathcal{L}\) is dense in \(\text{Alg} \mathcal{L}\) in the strong operator topology. As usual, \(\text{Alg} \mathcal{L}\) is the closed ideal of all bounded linear operators on \(X\) that leave every subspace in \(\mathcal{L}\) invariant. It is easy to see [ALL, p. 20] that \(\mathcal{L}\) has the strong rank one density property iff the strong closure of the set of all finite linear combinations of rank one elements in \(\mathcal{L}\) contains the identity operator on \(X\). In [ALL, Theorem 3.1] it was proved that every atomic Boolean subspace lattice having only two atoms on a Banach space has the strong rank one density property. Our goal in this section is to give an easier and (at least to us) more transparent proof of this result. The key is Lemma 2, which is a simple duality lemma for the strong rank one density property that is similar to the lemma proved earlier for nest algebras.

Lemma 2. Let \(\mathcal{A}\) be a collection of closed subspaces of a Banach space \(X\) such that the linear span of \(\bigcup \mathcal{A}\) is dense in \(X\) and for every \(Y\) in \(\mathcal{A}\), the intersection of \(Y\) with the closed linear span of \(\{W \in \mathcal{A} : W \neq Y\}\) is \(\{0\}\). Let \(\mathcal{O}\) be the collection of rank one bounded linear operators on
for $1 \leq X$ on a Banach space $A$. Theorem 5.1. Let $I_X$ be a linear operator on $X$ that leave every subspace in $A$ invariant. Then the identity operator $I_X$ on $X$ is not in the strong closure of the linear span of $O$ if and only if there is a finite rank bounded linear operator $S$ on $X$ having non zero trace such that for every $Y$ in $A$, the operator $S$ maps $Y$ into the closed linear span of $\{W \in A : W \neq Y\}$.

Proof. To go one way, assume that such an operator $S$ exists. The finite rank operator $S$ acts via trace duality as a strongly continuous linear functional on the space $B(X)$ of bounded linear operators on $X$ and, by the trace hypothesis, this linear functional is not zero at the identity operator. Thus it is sufficient to see that for every $x^* \otimes x$ in $O$ we have $\langle x^*, Sx \rangle = 0$. Now if $x^*$ is not identically zero on a subspace $Y$ in $A$, then $x$ must be in $Y$. This can happen for at most one $Y$ in $A$ by the intersection hypothesis on $A$, and must happen for at least one $Y$ in $A$ by the density hypothesis on $A$. But then for this $Y$ we have that $x^*$ is zero on the closed linear span of $\{W \in A : W \neq Y\}$, which contains $Sx$ by the hypothesis on $S$, which says that $\langle x^*, Sx \rangle = 0$.

Conversely, if $I_X$ is not in the strong closure of the linear span of $O$, then there is a strongly continuous linear functional on $B(X)$ that is non zero at $I_X$ and is zero on $O$. Being strongly continuous, this linear functional is given by trace duality with some bounded finite rank linear operator $S$ on $X$. Let $Y$ be in $A$ and let $Z$ be the closed linear span of $\{W \in A : W \neq Y\}$. Since $Y \cap Z = \{0\}$, $Z^\perp$ is total over $Y$. That is, given any non zero $y$ in $Y$, there is $y^*$ in $Z^\perp$ with $\langle y^*, y \rangle = 1$. The operator $y^* \otimes y$ is in $O$ and $\langle y^*, Sy \rangle = 0$ by the separation condition. Since this holds for every $y^*$ in $Z^\perp$, we deduce from the bipolar theorem that $Sy$ is in $Z$.

Theorem 5.1. Let $L$ be an atomic Boolean subspace lattice-ABSL-on a Banach space $X$ and assume that $L$ is generated by two atoms $A := \{X_1, X_2\}$. Then $L$ has the strong rank one density property.

Proof. If $L$ fails the strong rank one density property, then by Lemma 2 there is a finite rank bounded linear operator $S$ on $X$ having non zero trace such that $SX_1 \subseteq X_2$ and $SX_2 \subseteq X_1$. Lemma 2 applies because every subspace in an ABSL is the closed linear span of the atoms contained in the subspace. Since $X_1 + X_2$ is dense in $X$ and $S$ has finite rank, $SX = SX_1 + SX_2$; and $SX_1 \cap SX_2 \subseteq X_2 \cap X_1 = \{0\}$. Let $x_1, \ldots, x_k$ be a basis for $SX_1$ and let $x_{k+1}, \ldots, x_m$ be a basis for $SX_2$; hence $x_1, \ldots, x_m$ is a basis for $SX$. So we can write $S = \sum_{j=1}^m x_j^* \otimes x_j$ with each $x_j^*$ in $X^*$. Now $SX_2 \subseteq \text{span} x_1, \ldots, x_k$ and $x_1, \ldots, x_m$ are linearly independent, so $x_j^* \in X_j^*$ for $j < k$. Similarly, $x_j^* \in X_j^*$ for $1 \leq j \leq k$. So the trace of $S$ is zero, which is a contradiction.
Lemma 2 can also be used in the negative direction. For example, define three subspaces of $\ell_2$ by:

- $X_1 = \text{span}\{e_{2n} : n \in \mathbb{N}\}$
- $X_2 = \text{span}\{2^{-n}e_1 + e_{2n} + 4^{-n}e_{4n-1} : n \in \mathbb{N}\}$
- $X_3 = \text{span}\{4^{-n}e_1 + (4^{-n}e_1 + 2^{-n}e_{2n} + 8^{-n}e_{4n-1}) + 16^{-n}e_{4n-3} : n \in \mathbb{N}\}$

It is not difficult to check that $\mathcal{A} := \{X_1, X_2, X_3\}$ satisfies the hypotheses of Lemma 2. The collection $\mathcal{A}$ does not form the atoms of an atomic Boolean subspace lattice on $\ell_2$. One can see this by verifying that

$$e_1 \in (X_1 + X_2) \cap (X_1 + X_3) \cap (X_2 + X_3)$$

The rank one operator $S := e_1 \otimes e_1$ satisfies the conditions on $S$ in Lemma 2.

**Remark 5.2.** In [LaLo], Lambrou and Longstaff give an example of an ABSL that has four atoms but fails the strong rank one density property. They also refer to a preprint by Lambrou and Spanoudakis that gives an example of an ABSL that has three atoms but fails the strong rank one density property, but it seems that this was never published. The argument in [LaLo] is a separation argument. The authors construct a rank two operator that separates the identity from $\mathcal{A}lg(\mathcal{L})$, but check directly that the trace of the operator composed with rank one elements from $\mathcal{A}lg(\mathcal{L})$ is zero. The computations are tedious, but it is unfortunately not much less tedious to verify instead that the rank two operator they construct satisfies the hypothesis of Lemma 2.

**References**


Institute of Mathematics, The Polish Academy of Sciences
E-mail address: t.figiel@impan.gda.pl

Department of Mathematics, Texas A&M University, College Station, TX 77843–3368 U.S.A
E-mail address: johnson@math.tamu.edu