REPRESENTING COMPLETELY CONTINUOUS OPERATORS THROUGH WEAKLY ∞-COMPACT OPERATORS

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ABSTRACT. Let $V, W_\infty,$ and $W$ be operator ideals of completely continuous, weakly $\infty$-compact, and weakly compact operators, respectively. We prove that $V = W_\infty \circ W^{-1}$. As an immediate application, the recent result by Dowling, Freeman, Lennard, Odell, Randrianantoanina, and Turett follows: the weak Grothendieck compactness principle holds only in Schur spaces.

1. Introduction

Let $L, K, W,$ and $V$ denote the operator ideals of bounded linear, compact, weakly compact, and completely continuous operators. Let $X$ and $Y$ be Banach spaces. Recall that a linear map $T: X \to Y$ is completely continuous, i.e., $T \in V(X,Y)$, if $T$ takes weakly null sequences in $X$ to null sequences in $Y$.

Recall that $K \subset V$ and $K \subset W$, but $V$ and $W$ are incomparable [9, 1.11.8]. The starting point for the present note was the following well-known formula [9, 3.2.3]:

$$V = K \circ W^{-1}.$$ 

Recall that the right-hand quotient $A \circ B^{-1}$ of two operator ideals $A$ and $B$ is the operator ideal that consists of all operators $T \in L(X,Y)$ such that $TS \in A(Z,Y)$ whenever $S \in B(Z,X)$ for some Banach space $Z$ [9, 3.1.1].

Let $(x_n) \subset X$ be a bounded sequence. It is well known and easy to see that $(x_n)$ defines an operator $\Phi_{(x_n)} \in L(\ell_1, X)$ through the equality

$$\Phi_{(x_n)}(a_k) = \sum_{k=1}^{\infty} a_k x_k; \quad (a_k) \in \ell_1.$$ 

The main tool in the proof of the formula $V = K \circ W^{-1}$ in [9, 3.2.3] is the simple fact that $\Phi_{(x_n)}: c_0^* \to X$ is weak$^*$-to-weak continuous if $(x_n)$ is weakly null.

Let $B_X$ denote the closed unit ball of $X$. A subset $K$ of $X$ is called relatively weakly $\infty$-compact if $K \subset \Phi_{(x_n)}(B_{\ell_1})$ for some weakly null sequence $(x_n)$ in $X$. An operator $T \in L(X,Y)$ is weakly $\infty$-compact if $T(B_X)$ is a relatively weakly $\infty$-compact subset of $Y$. Weakly $\infty$-compact (more generally, weakly $p$-compact) operators were considered by Castillo and Sanchez [4] in 1993 and by Sinha and Karn [10] in 2002 (for an even more general version of weakly $(p,r)$-compact operators, see [3]).

Denote by $W_\infty$ the class of all weakly $\infty$-compact operators acting between arbitrary Banach spaces. An easy straightforward verification (as
in [1, Propositions 2.1 and 2.2]) shows that $\mathcal{W}_\infty$ is a surjective operator ideal. The main result of this note reads as follows.

**Theorem 1.1.** $\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1}$.

An immediate consequence is that the weak Grothendieck compact principle holds only in Schur spaces. Recall that $X$ has the *Schur property* (is a Schur space) if weakly null sequences in $X$ are norm null.

**Corollary 1.2.** [7, Theorem 1] Every weakly compact subset of a Banach space $X$ is contained in the closed convex hull of a weakly null sequence if and only if $X$ has the Schur property.

Our method of proof relies on the Davis–Figiel–Johnson–Pełczyński factorization theorem [5], providing also an alternative proof for the Dowling–Freeman–Lennard–Odell–Randrianantoanina–Turett theorem [7], where Schauder basis theory was used.

2. Proof of Theorem 1.1

The following fact is well known; we include a proof for completeness.

**Proposition 2.1.** If $(x_n)$ is a weakly null sequence in a Banach space $X$, then $\Phi_{(x_n)}(B_{\ell_1})$ is weakly compact and coincides with the closed absolutely convex hull of $(x_n)$.

**Proof.** (cf. [2, proof of the “if” part of Theorem 3]). The set $\Phi_{(x_n)}(B_{\ell_1})$ is clearly absolutely convex. It is also weakly compact because $\Phi_{(x_n)} : c_0^* \to X$ is weak*-to-weak continuous and $B_{\ell_1} = B_{c_0}^*$ is weak* compact. Hence, $\Phi_{(x_n)}(B_{\ell_1})$ is a closed absolutely convex subset of $X$ containing $(x_n)$. Since $\Phi_{(x_n)}(B_{\ell_1})$ is obviously contained in the closed absolutely convex hull of $(x_n)$, it coincides with the latter set.

Let $X$ be a Banach space. By the Grothendieck compactness principle, any compact subset of $X$ is contained in $\Phi_{(x_n)}(B_{\ell_1})$ for some null sequence $(x_n)$ in $X$. Therefore, the relatively compact sets are relatively weakly $\infty$-compact, and we get from Proposition 2.1 the following (known) result.

**Corollary 2.2.** $\mathcal{K} \subset \mathcal{W}_\infty \subset \mathcal{W}$.

From the proof of Proposition 3.1 below, it can be seen that these inclusions are strict.

Since $\mathcal{K} \subset \mathcal{W}_\infty$, we clearly have that $\mathcal{K} \circ \mathcal{W}^{-1} \subset \mathcal{W}_\infty \circ \mathcal{W}^{-1}$. Since also $\mathcal{V} \subset \mathcal{K} \circ \mathcal{W}^{-1}$ (this is the obvious “part” of the equality $\mathcal{V} = \mathcal{K} \circ \mathcal{W}^{-1}$),

$\mathcal{V} \subset \mathcal{W}_\infty \circ \mathcal{W}^{-1}$.

For the **proof of Theorem 1.1**, it remains to show that

$\mathcal{W}_\infty \circ \mathcal{W}^{-1} \subset \mathcal{V}$.

**Proof.** Let $X$ and $Y$ be Banach spaces and let $T \in \mathcal{W}_\infty \circ \mathcal{W}^{-1}(X,Y)$. Assume for contradiction that $T \not\in \mathcal{V}(X,Y)$. Then there exists a weakly null sequence $(x_n)$ in $X$ such that $(Tx_n)$ is not a null sequence in $Y$. Passing to a subsequence of $(x_n)$, we may assume that $\|Tx_n\| \geq \delta, n \in \mathbb{N}$, for some $\delta > 0$. Hence, $(Tx_n)$ is not relatively compact.
Since $\Phi(x_n) \in W(\ell_1, X)$ (see Proposition 2.1), by the Davis–Figiel–Johnson–Pełczyński factorization theorem [5], there exist a reflexive space $R$ and weakly compact operators $\Phi : \ell_1 \to R$ with $\|\Phi\| = 1$ and $J : R \to X$ such that $\Phi(x_n) = J\Phi$. From the definition of $W_\infty \circ W^{-1}$, we get that $TJ \in W_\infty(R,Y)$ because $T \in W_\infty \circ W^{-1}(X,Y)$ and $J \in W(R,X)$. Hence, there exists a weakly null sequence $(y_n)$ in $Y$ such that $TJ(B_R) \subset \Phi(y_n)(B_{\ell_1})$. In particular, $Tx_n = TJ(x_n)e_n = TJ\Phi e_n \in \Phi(y_n)(B_{\ell_1})$, $n \in \mathbb{N}$, where $(e_n)$ is the unit vector basis in $\ell_1$.

Denote by $\overline{\Phi}(y_n)$ the injective associate of $\Phi(y_n)$, which means that $\Phi(y_n) = \overline{\Phi}(y_n)q$, where $q : \ell_1 \to Z := \ell_1/\ker \Phi(y_n)$ is the quotient mapping. Since ran $TJ \subset$ ran $\Phi(y_n) = \ker \overline{\Phi}(y_n)$, we can consider the linear operator $\overline{\Phi}(y_n)^{-1}TJ : R \to Z$. This operator is bounded: if $r \in B_R$, then $TJr = \Phi(y_n)\alpha$ for some $\alpha \in B_{\ell_1}$ and $\|\overline{\Phi}(y_n)^{-1}TJr\| = \|q\alpha\| \leq 1$.

We claim that $Z$ has the Schur property. Indeed, by Grothendieck’s result [8, Theorem 10] (see also Remark 2.3 below), the dual $W^*$ of any closed subspace $W$ of $c_0$ has the Schur property. It remains to observe that ker $\Phi(y_n)$ is weak* closed in $\ell_1 = c_0'$ (because $\Phi(y_n)$ is weak*-to-weak continuous), hence ker $\Phi(y_n) = W^\perp$ for some closed subspace $W$ of $c_0$, and $Z = W^*$.

Since $R$ is reflexive and $Z$ has the Schur property, $\mathcal{L}(R,Z) = \mathcal{K}(R,Z)$. In particular, $\overline{\Phi}(y_n)^{-1}TJ$ and therefore also $\overline{\Phi}(y_n)^{-1}\overline{\Phi}(y_n)^{-1}TJ = TJ$ are compact operators. It follows that $(Tx_n) = (TJ\Phi e_n) \subset (TJ)(B_R)$ is relatively compact, a contradiction that completes the proof of Theorem 1.1.

**Remark 2.3.** Let $W$ be a closed subspace of $c_0$. To prove that the dual $W^*$ has the Schur property, Grothendieck [8, Theorem 10] first establishes that $W$ has the Dunford–Pettis property (DPP). Grothendieck’s easy and beautiful proof can be found in Diestel’s survey article [6, pages 25–26, see also Theorem 4]. Since $W$ does not contain a copy of $\ell_1$, relying on Rosenthal’s $\ell_1$ theorem, Diestel [6, Theorem 3] quickly concludes that $W^*$ has the Schur property. Let us provide a version of Grothendieck’s proof [8, pages 171–172], showing that the DPP of $W$ implies that $W^*$ has the Schur property.

The DPP of $W$ means that every weakly compact operator with domain $W$ is completely continuous, hence compact (because $W^*$ is separable). It follows easily that then $W^*$ has the Schur property: given $(w_n^*)$ weakly null in $W^*$, consider the weakly compact operator $S : W \to c_0$ defined by $Sw = (\langle w_n^*, w \rangle)$, and use that $S^* = \Phi(w_n^*)$ is a compact operator.

### 3. Applications of Theorem 1.1

It is well known that $\mathcal{K} \subset \mathcal{V}$. As we see now, $W_\infty$ lies strictly between $\mathcal{K}$ and $\mathcal{V}$.

**Proposition 3.1.** $\mathcal{K} \subset W_\infty \subset \mathcal{V}$ and both inclusions are strict.

**Proof.** It is obvious that $\mathcal{A} \subset \mathcal{A} \circ B^{-1}$ for any two operator ideals $\mathcal{A}$ and $\mathcal{B}$. Hence, $W_\infty \subset W_\infty \circ W^{-1} = \mathcal{V}$ by Theorem 1.1, and the inclusion $\mathcal{K} \subset W_\infty$ was observed in Corollary 2.2.

To see that $\mathcal{K} \neq W_\infty$, consider the identity embedding $j : \ell_1 \to c_0$ that is not compact but is weakly $\infty$-compact, because $j = \Phi(e_n)$, where $(e_n)$ is the
unit vector basis of \( c_0 \). On the other hand, the identity operator on \( \ell_1 \) is completely continuous (because \( \ell_1 \) has the Schur property) but since it is not weakly compact, it is not weakly \( \infty \)-compact either (recall that \( \mathcal{W}_\infty \subset \mathcal{W} \)).

(After another way to see that \( \mathcal{W}_\infty \neq \mathcal{V} \) is to use that \( \mathcal{W}_\infty = \mathcal{W}_\infty^{sur} \), the surjective hull, but \( \mathcal{V} \neq \mathcal{V}^{sur} = \mathcal{L} \). Now it is easy to see that the inclusion \( \mathcal{W}_\infty \subset \mathcal{W} \) in Corollary 2.2 is strict: the identity operator on \( \ell_2 \) is weakly compact but since it is not completely continuous, it is not weakly \( \infty \)-compact.

Let \( X \) and \( Y \) be Banach spaces and \( T \in \mathcal{L}(X,Y) \). It is well known (and clear thanks to the Eberlein–Šmulian theorem) that \( T \in \mathcal{V}(X,Y) \) if and only if \( T \) takes relatively weakly compact subsets of \( X \) into relatively weakly \( \infty \)-compact subsets of \( Y \).

**Theorem 3.2.** Let \( X \) and \( Y \) be Banach spaces and \( T \in \mathcal{L}(X,Y) \). Then \( T \in \mathcal{V}(X,Y) \) if and only if \( T \) takes relatively weakly compact subsets of \( X \) into relatively weakly \( \infty \)-compact subsets of \( Y \).

**Proof.** The “only if” part is obvious because relatively compact sets are relatively weakly \( \infty \)-compact. From the definition of \( \mathcal{W}_\infty \circ \mathcal{W}^{-1} \), it is clear that if \( T \) takes relatively weakly compact sets into relatively weakly \( \infty \)-compact sets, then \( T \in \mathcal{W}_\infty \circ \mathcal{W}^{-1}(X,Y) \). By Theorem 1.1, this means that \( T \in \mathcal{V}(X,Y) \).

Let \( \mathcal{A} \) be an operator ideal. Recall that the space ideal \( \text{Space}(\mathcal{A}) \) is defined as the class of all Banach spaces \( X \) such that the identity operator on \( X \) belongs to \( \mathcal{A}(X,X) \). If \( \mathcal{A} \) and \( \mathcal{B} \) are operator ideals, then obviously \( X \in \text{Space}(\mathcal{A} \circ \mathcal{B}^{-1}) \) if and only if \( \mathcal{B}(Z,X) \subset \mathcal{A}(Z,X) \) for all Banach spaces \( Z \).

From the definitions, it is clear that \( \text{Space} (\mathcal{V}) \) is the class of all Banach spaces with the Schur property. Theorem 1.1 yields that

\[
\text{Space} (\mathcal{V}) = \text{Space} (\mathcal{W}_\infty \circ \mathcal{W}^{-1}).
\]

This can be reformulated as follows. Note that the equivalence \( (a) \iff (b) \) below is precisely Corollary 1.2 and, as was mentioned in the Introduction, it is due to [7, Theorem 1].

**Theorem 3.3.** For a Banach space \( X \), the following statements are equivalent:

1. \( X \) has the Schur property;
2. the weak Grothendieck compactness principle holds in \( X \);
3. \( \mathcal{W}(Z,X) \subset \mathcal{W}_\infty(Z,X) \) for all Banach spaces \( Z \).

**Proof.** We already observed that \( (a) \iff (c) \) thanks to Theorem 1.1. The implications \( (a) \Rightarrow (b) \Rightarrow (c) \) are obvious. (But \( (a) \iff (b) \) is also the special case of Theorem 3.2, when \( T \) is the identity operator on \( X \).)

**Remark 3.4.** By the Davis–Figiel–Johnson–Pełczyński factorization theorem, \( (c) \) is equivalent to

1. all injective operators from reflexive Banach spaces to \( X \) are weakly \( \infty \)-compact.
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