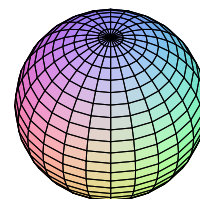


Instructions. Do your choice of **five** of the following six problems. These are essay questions. To obtain maximal credit, show your work and explain your reasoning.

1. Sphere of influence You know that a sphere of radius r has volume equal to $\frac{4}{3}\pi r^3$ and surface area equal to $4\pi r^2$. The goal of this problem is to demonstrate that you can prove these formulas using methods from this course.



- (a) Viewing the sphere as a solid of revolution, set up an integral that represents the volume of the sphere.
(You may use either the disk method or the shell method.)
- (b) Viewing the surface of the sphere as a surface of revolution, set up an integral that represents the surface area of the sphere.
- (c) You need not carry out the complete evaluation of the integrals that you set up in parts (a) and (b), but indicate how you would proceed in each case.
(For example, if you would integrate by parts, say what you would choose for u and v ; if you would make a substitution, say what that substitution would be.)

2. A Taylor-made problem Use your knowledge of power series to show that the following evaluations of numerical series are correct. Do your choice of **two** of the three parts.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \pi^{2n}}{(2n)!} = 2.$ (The series starts $\frac{\pi^2}{2!} - \frac{\pi^4}{4!} + \frac{\pi^6}{6!} - \dots$.)

(b) $\sum_{n=1}^{\infty} \frac{(\ln 3)^n}{n!} = 2.$ (The series starts $(\ln 3) + \frac{(\ln 3)^2}{2!} + \frac{(\ln 3)^3}{3!} + \dots$.)

(c) $\sum_{n=3}^{\infty} \frac{n}{2^{n-1}} = 2.$ (The series starts $\frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \dots$.)

Colin
Maclaurin



(1698–1746)

3. Just the factorial please Leonhard Euler (pronounced “oiler”), shown here on a Swiss bank note, was one of the most prolific and influential mathematicians not only of the eighteenth century but of all time.



One of Euler’s many contributions is the *gamma function* $\Gamma(p)$, which when p is a positive real number can be defined by an improper integral:

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx. \quad (\dagger)$$

- Show that $\Gamma(1) = 1$ by computing the improper integral $\int_0^{\infty} e^{-x} dx$.
- Accepting for the moment that the improper integral in equation (\dagger) above does converge when $p > 0$, use integration by parts to show that $\Gamma(p + 1) = p\Gamma(p)$.
- Using the preceding two parts, deduce the numerical values of $\Gamma(2) = \Gamma(1 + 1)$, $\Gamma(3) = \Gamma(2 + 1)$, and $\Gamma(4) = \Gamma(3 + 1)$. Can you now see what the value of $\Gamma(n + 1)$ is when n is a positive integer?
- Optional, for extra credit:** Prove that the improper integral in equation (\dagger) above really does converge when $p > 0$.
(Caution: when $0 < p < 1$, the integral is improper at both ends.)

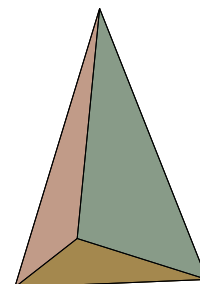
Remark: In a previous examination, you encountered Euler’s constant γ (lowercase Greek letter gamma). A connection between Γ and γ is the infinite product representation

$$\Gamma(p) = \frac{e^{-\gamma p}}{p} \prod_{n=1}^{\infty} \frac{ne^{p/n}}{n+p} \quad \text{when } p > 0.$$

4. A Platonic relationship The thirteenth book of Euclid's *Elements* (circa 300 BC) studies the five regular polyhedra: the tetrahedron, the octahedron, the cube, the dodecahedron, and the icosahedron. These five solids are known as the Platonic solids.

This problem concerns the tetrahedron whose vertices are the four points in three-dimensional space having coordinates $(0, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$. Each three of these four points constitute the vertices of an equilateral triangle, and those four triangles are the faces of the tetrahedron.

Determine the following quantities. (Notice that by symmetry, it will not matter which vertex or edge or face you start with.)



- the length of an edge of the tetrahedron
- the angle between two edges that meet at a vertex
- the angle between two faces
(warning: the answer is not the same as the previous answer)
- the distance between a face and the vertex that is not in that face

5. Nothing succeeds like success If you ask Maple to compute the integral

$$\int_0^1 \arctan \left(\sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} \right) dx, \quad (\#)$$

you will find that Maple gives up and returns no answer. Demonstrate that you can succeed where Maple fails by doing your choice of **one** of the following two parts.

- Show by an exact computation that the integral $(\#)$ equals $\frac{1}{2}$.
(Hint: if you use the double-angle identities, a miraculous simplification occurs.)
- Use approximation methods to compute the integral $(\#)$ numerically with an error of less than 10%.



6. “It’s my own invention” [The White Knight in *Through the Looking Glass* by Lewis Carroll]

Make up and solve **one** interesting problem that involves **two** of the following topics.



- (a) the fundamental theorem of calculus
 - (b) work
 - (c) average value
 - (d) partial fractions
 - (e) differential equations
 - (f) arc length
 - (g) center of mass
 - (h) hydrostatic pressure
 - (i) the integral test for convergence of infinite series
 - (j) the ratio test
 - (k) the binomial series
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