Math 152 (honors sections)

Harold P. Boas
boas@tamu.edu

New power series from old

The geometric series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

has radius of convergence equal to 1.

We can get new series expansions via

- substitution;
- differentiation;
- integration.

Differential equations and series

Example. Solve $y' = y$ with initial condition $y(0) = 1$.

Method 1. Separate variables to get $\frac{dy}{y} = dx$. Then $\ln|y| = x + C$. By the initial condition, $C = 0$. So $y = e^x$.

Method 2. Use a series. Set

$$y = 1 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + \cdots$$

Then

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$$

Matching coefficients gives $a_1 = 1$, $a_2 = \frac{1}{2}a_1 = \frac{1}{2}$, $a_3 = \frac{1}{3}a_2 = \frac{1}{2\cdot 3}$, $a_n = \frac{1}{n}a_{n-1} = \frac{1}{n!}$. Thus

$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \tfrac{1}{n!}x^n + \cdots .$$

The series converges for all x.

Using series to approximate integrals

Example: Find $\int_0^{1/2} \frac{1}{1+x^6}\,dx$ correct to six decimal places (#26, p. 622).

Solution. Since $\frac{1}{1+x^6} = 1 - x^6 + x^{12} - x^{18} + \cdots$, the integral equals

$$\frac{1}{2} - \frac{1}{7}\cdot\frac{1}{2^7} + \frac{1}{13}\cdot\frac{1}{2^{13}} - \frac{1}{19}\cdot\frac{1}{2^{19}} + \cdots .$$

By the alternating series remainder estimate, the remainder after the first three terms is at most $\frac{1}{19}\cdot\frac{1}{2^{19}} \approx 1.0\times 10^{-7}$, so

$$\int_0^{1/2} \frac{1}{1+x^6}\,dx \approx \frac{1}{2} - \frac{1}{7}\cdot\frac{1}{2^7} + \frac{1}{13}\cdot\frac{1}{2^{13}} \approx 0.498893$$

correct to six decimal places.

Homework

- Read section 10.6, pages 618–622.
- Do the Suggested Homework exercises for section 10.6.
- Do exercise 34 on page 623 in section 10.6.