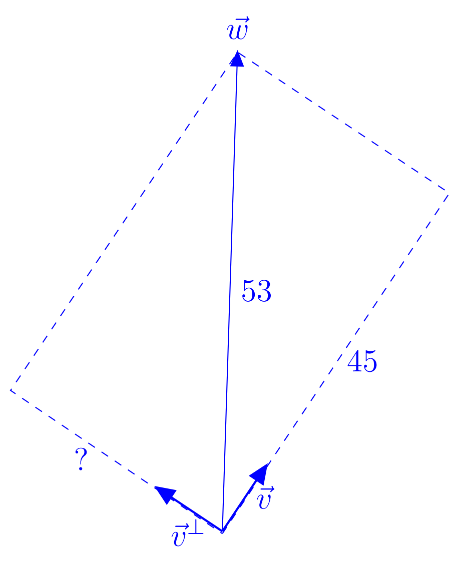
Calculus

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Suppose vector $\vec{v} = 2\vec{\imath} + 3\vec{\jmath}$ and vector $\vec{w} = a\vec{\imath} + b\vec{\jmath}$ for some numbers a and b. If the length $|\vec{w}| = 53$, and the scalar projection of $\vec{w}$ onto $\vec{v}$ equals 45, find the scalar projection of $\vec{w}$ onto $\vec{v}^{\perp}$, the orthogonal complement of $\vec{v}$.
[Recall that the scalar projection of $\vec{w}$ onto $\vec{v}$ is the component of $\vec{w}$ in the direction of $\vec{v}$, that is, the magnitude of the vector projection of $\vec{w}$ onto $\vec{v}$. In two-dimensional space, the orthogonal complement of a vector is the vector obtained by a 90° counterclockwise rotation.]

Solution. If you attempt this problem by using formulas for the vector projection and the scalar projection, then you are working too hard. The length of a vector is the square root of the sum of the squares of the components, and this equality holds for the components of the vector with respect to any orthogonal coordinate frame (by the Pythagorean theorem for right triangles).



See the figure, which makes it evident that the unknown quantity equals $\sqrt{53^2 - 45^2}$, or $\sqrt{784}$, or 28.

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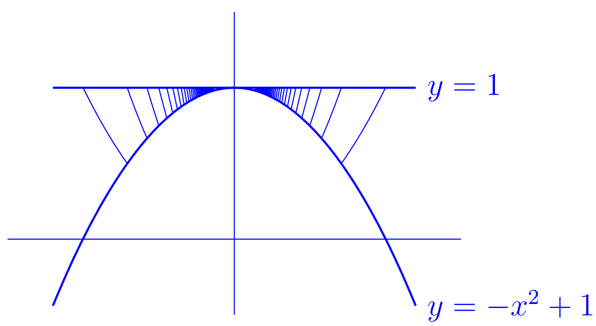
2. Recall that the floor function $\lfloor x \rfloor$ is the greatest integer less than or equal to x. (The textbook denotes this function by the symbol $[\![x]\!]$ and calls it the greatest integer function.) Use the Squeeze Theorem to prove that
$$\lim_{x\to 0}\left(x^2\left\lfloor \frac{1}{x^2}\right\rfloor\right) = 1.$$
[Neither the TI-89 calculator nor Maple can do this limit!]

Solution. The floor of a number does not exceed the number and is at most 1 less than the number. Thus
$$-1+\frac{1}{x^2} \le \left\lfloor \frac{1}{x^2}\right\rfloor \le \frac{1}{x^2}, \qquad \text{so} \qquad -x^2+1 \le x^2\left\lfloor \frac{1}{x^2}\right\rfloor \le 1.$$
The upper and lower bounds 1 and $-x^2+1$ both have limit 1 as $x \to 0$, so the Squeeze Theorem implies that the middle quantity $x^2\lfloor 1/x^2\rfloor$ has limit 1 too. See the figure for a visual interpretation: the graph in the middle represents the discontinuous function $x^2\lfloor 1/x^2\rfloor$.



3. In February 2006, students at Michigan Tech rolled a world-record snowball, a sphere 6.75 feet in diameter. If the snowball subsequently melted at a rate of 1 cubic foot per day, how fast was the radius of the snowball changing, in units of inches per day, when the radius was equal to 20 inches?
[One foot equals 12 inches, so one cubic foot equals 1728 cubic inches. The volume of a sphere of radius r equals $\frac{4}{3}\pi r^3$.]

Solution. Since the volume $V = \frac{4}{3}\pi r^3$, differentiating with respect to time t gives $dV/dt = 4\pi r^2\, dr/dt$. It is given that $dV/dt = -1728$ cubic inches per day, so

$$\frac{dr}{dt} = -\frac{1728}{4\pi r^2}.$$

Evaluating this expression when $r = 20$ shows that

$$\frac{dr}{dt} = -\frac{1728}{1600\pi} = -\frac{27}{25\pi} \approx -0.344 \text{ inches per day.}$$

4. The equation $e^y + \ln(x) = 0$ implicitly determines y as a function $f(x)$ when $0 < x < 1$. Show that the graph of f has an inflection point where $x = 1/e$.

   Solution. Differentiating implicitly shows that $e^y y' + x^{-1} = 0$, so the first derivative $y' = -e^{-y}/x$. Differentiating a second time using the quotient rule shows that $y'' = -[-xe^{-y}y' - e^{-y}]/x^2 = (xy' + 1)e^{-y}/x^2$. Substituting the already computed value for y' into this expression shows that $y'' = (1 - e^{-y})e^{-y}/x^2$. Since the factor e^{-y}/x^2 is always positive, the sign of the second derivative y'' is the same as the sign of $1 - e^{-y}$. Therefore y'' is positive when $y > 0$, and y'' is negative when $y < 0$. Hence there is an inflection point at the point on the curve where $y = 0$. From the original equation for the curve, one sees that $y = 0$ when $\ln(x) = -1$, which means that $x = e^{-1}$.

   Alternatively, one can compute the second derivative directly by first solving the original equation to see that $y = \ln\ln(1/x)$.

5. For how many values of x does the slope of the graph of $\tan^{-1}(x)$ equal the slope of the graph of $1/\tan(x)$? Explain how you know.
   [Recall that $\tan^{-1}(x)$ and $1/\tan(x)$ are different functions, because the exponent -1 means inverse function, not reciprocal.]

   Solution. The derivative of $\tan^{-1}(x)$ equals $1/(1+x^2)$, and the derivative of $1/\tan(x)$ can be obtained by applying the quotient rule to $\cos(x)/\sin(x)$ to get $[-\sin^2(x) - \cos^2(x)]/\sin^2(x) = -1/\sin^2(x)$. Since $1/(1 + x^2)$ is always positive, while $-1/\sin^2(x)$ is always negative (or undefined), there is *no* value of x at which the original two graphs have the same slope.

6. Determine a number c and a function f such that $3\int_0^x f(t)\,dt = e^{3x} - c$.

   Solution. When $x = 0$, the left-hand side equals 0, while the right-hand side equals $1 - c$. Therefore $c = 1$. Taking the derivative of both sides using the fundamental theorem of calculus shows that $3f(x) = 3e^{3x}$. Therefore $f(x) = e^{3x}$.

7. Suppose that a particle moves in a circle and has position vector equal to $(\cos t, \sin t)$ at time t. Show that the corresponding velocity vector is orthogonal to the acceleration vector for every t.
   [This is a standard property of circular motion that you may have learned in physics class.]

   Solution. Taking the derivative of the position vector gives the velocity vector $(-\sin t, \cos t)$. Taking another derivative gives the acceleration vector $(-\cos t, -\sin t)$. The second vector is the orthogonal complement of the first; the dot product of the two vectors equals $(-\sin t)(-\cos t) + (\cos t)(-\sin t)$, which is 0.

8. Give a careful statement of **one** of the following definitions:

   (a) the precise definition of limit (using ϵ and δ);

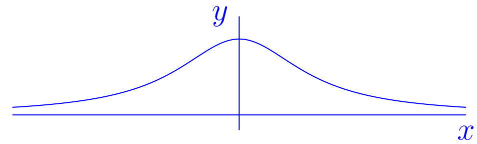
   (b) the definition of derivative (as a limit);

   (c) the definition of the definite integral (as a limit of Riemann sums).

   Solution. These definitions can be found in the textbook on pages 102, 148, and 378.

9. Sketch an example of a graph having no points of discontinuity, exactly one critical point, and exactly two inflection points.

   Solution. There are many examples. The graph shown below is the graph of the function $1/(1 + x^2)$.



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10. **Optional problem for extra credit**

Abby, Blake, and Cory are trying to compute the indefinite integral $\int 2\sin(x)\cos(x)\,dx$.

Abby says, "Substituting $u = \sin(x)$ and $du = \cos(x)\,dx$, I get

$$\int 2\sin(x)\cos(x)\,dx = \int 2u\,du = u^2,$$

so back substituting gives the answer $\sin^2(x)$."

Blake says, "I remember a trig identity $2\sin(x)\cos(x) = \sin(2x)$, so substituting $u = 2x$ and $du = 2\,dx$, I get

$$\int 2\sin(x)\cos(x)\,dx = \int \sin(2x)\,dx = \tfrac{1}{2}\int \sin(u)\,du = -\tfrac{1}{2}\cos(u),$$

and back substituting gives the answer $-\frac{1}{2}\cos(2x)$."

Cory says, "I tried this problem on my TI-89 calculator, and it gave me the answer $-(\cos(x))^2$."

Who (if anyone) is right, who is wrong, and what mistakes did these students make?

Solution. All three students are equally right and equally wrong. All three students forgot to add an arbitrary constant to the anti-derivative; otherwise, their results are all correct. Abby's answer should really be $\sin^2(x) + c_1$, Blake's answer should be $-\frac{1}{2}\cos(2x) + c_2$, and Cory's answer should be $-(\cos(x))^2 + c_3$.

Since $\sin^2(x) + \cos^2(x) = 1$, Abby and Cory have equivalent answers but with different arbitrary constants (namely, $c_3 = 1 + c_1$). Since $\cos(2x) = 1 - 2\sin^2(x)$, Blake's answer is equivalent to $-\frac{1}{2} + \sin^2(x) + c_2$, and this agrees with Abby's answer but with a different name for the arbitrary constant (namely, $c_1 = -\frac{1}{2} + c_2$).