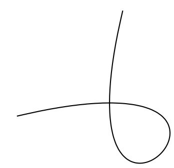
Math 171-501 Quiz 7 Fall 2007

Calculus

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. A curve in the xy-plane is given in parametric form by $x(t) = t^3 - 3t^2$ and $y(t) = t^3 - 3t$, where the parameter t runs through the real numbers. Find the points on the curve where the tangent line is vertical (that is, parallel to the y-axis).
   [This is exercise 12 on page 214 of the textbook.]



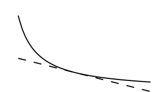
   Solution. According to the chain rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

   To find points where the tangent line is vertical (undefined slope), we should find points where the denominator $dx/dt = 0$. Since $dx/dt = 3t^2 - 6t = 3t(t-2)$, the relevant values of t are 0 and 2. Since $x(0) = 0$ and $y(0) = 0$, while $x(2) = -4$ and $y(2) = 2$, the corresponding points on the curve are $(0, 0)$ and $(-4, 2)$.

   [Incidentally, the second of these points happens to be the point of self-intersection of the curve. The second branch of the curve goes through the point $(-4, 2)$ with slope 0 when $t = -1$.]

2. Find the linear approximation of the function $\dfrac{1}{2+x}$ at the point $a = 0$.



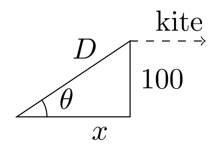
   Solution. The linear approximation formula says that $f(x) \approx f(a) + f'(a)(x - a)$. Taking $f(x)$ equal to $1/(2 + x)$ and a equal to 0, we find that $f(0) = 1/2$ and $f'(x) = -1/(2 + x)^2$, so $f'(0) = -1/4$. Thus

$$\frac{1}{2+x} \approx \frac{1}{2} - \frac{1}{4}x \qquad \text{for } x \text{ close to } 0.$$

   In the book's notation, $L(x) = \frac{1}{2} - \frac{1}{4}x$. Another way to think about this answer is that the tangent line to the curve $y = 1/(2 + x)$ at the point where $x = 0$ is $y = \frac{1}{2} - \frac{1}{4}x$.

Calculus

3. A kite 100 feet above the ground moves horizontally at a speed of 8 feet/second. At what rate is the angle between the string and the horizontal decreasing when 200 feet of string have been let out?
[This is exercise 22 on page 220 of the textbook.]



Solution.

(a) The given information is that $dx/dt = 8$.

(b) What we are supposed to find is $d\theta/dt$ when $D = 200$.

(c) From the diagram, we can read off the relation $x = 100\cot\theta$.

(d) Differentiating the relation with respect to t using the chain rule, we find that $dx/dt = 100(-\csc^2\theta)\dfrac{d\theta}{dt}$.

(e) From the diagram, you can see that when $D = 200$, the value of $\csc\theta$ is $200/100$ or 2. Plugging the numbers into the equation from the preceding step, we have that $8 = 100 \times (-4) \times \dfrac{d\theta}{dt}$. Thus $\dfrac{d\theta}{dt} = -1/50$ or -0.02. In words, the angle is decreasing at a rate of 0.02 radians per second when $D = 200$.

Remarks: When $D = 200$, the angle $\theta = \pi/6$ and $x = 100\sqrt{3}$, but you do not actually need either of those pieces of information to solve the problem.

If in step (c) you wrote the relation $\dfrac{100}{x} = \tan\theta$, then your differentiated relation would be $-\dfrac{100}{x^2}\dfrac{dx}{dt} = \sec^2\theta\dfrac{d\theta}{dt}$. Plugging in the numbers will lead to the same final answer as above.