Calculus

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. Find an approximate solution of the equation $x^5 - x^2 - 32 = 0$ by doing one iteration of Newton's method starting from the initial guess $x_0 = 2$.

   Solution. Here $f(x) = x^5 - x^2 - 32$, so $f'(x) = 5x^4 - 2x$. According to Newton's method,

   $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-4}{76} = 2 + \frac{1}{19} \approx 2.05,$$

   rounded to two decimal places.

   Remark: More iterations of Newton's method yield the approximate solution $x \approx 2.04996956$, correct to eight decimal places. Thus the first iteration already gives the answer correct to two decimal places (after rounding).

2. Find an equation for the line tangent to the graph of $y = e^{\sin x}$ at the point on the graph where $x = 0$.

   Solution. By the chain rule, $dy/dx = e^{\sin x} \cos x$. Therefore the slope when $x = 0$ is equal to $e^{\sin 0} \cos 0$, or 1. Since $y = 1$ when $x = 0$, an equation for the tangent line is $y - 1 = x$.

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3. For the curve given in parametric form by $x(t) = \ln(2t)$ and $y(t) = e^{3t}$, find the slope dy/dx at the point on the curve where $t = 1$.

   Solution. The chain rule tells us that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

   Now $dy/dt = 3e^{3t}$ by the chain rule. Since $x(t) = \ln(2t) = \ln 2 + \ln t$, we have $dx/dt = 1/t$. Therefore

$$\frac{dy}{dx} = 3te^{3t},$$

   and the slope when $t = 1$ is equal to $3e^3$.

4. The TI-89 calculator says that

$$\frac{d}{dx}\left(x^{1/x}\right) = \left(\frac{1}{x^2} - \frac{\ln(x)}{x^2}\right) x^{1/x}.$$

   Supply a calculation that confirms this result, assuming that $x > 0$. (Use the method of logarithmic differentiation.)

   Solution. If $y = x^{1/x}$, then $\ln y = \frac{1}{x}\ln x$. Differentiating using the chain rule on the left-hand side and the product rule on the right-hand side shows that

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{1}{x} \cdot \frac{d}{dx} \ln x + (\ln x) \cdot \frac{d}{dx}\left(\frac{1}{x}\right) \\ &= \frac{1}{x^2} - \frac{\ln x}{x^2}, \end{aligned}$$

   since the derivative of $\ln x$ is $1/x$, and the derivative of $1/x$ is $-1/x^2$. Multiplying the equation by y, which equals $x^{1/x}$, leads to the result from the calculator.