Calculus

Math 172-502 Second Examination Fall 2005

Work all five problems. These are essay questions. To obtain maximal credit, show your work and explain your reasoning.

1. Show both that the improper integral $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ converges and that the value of the integral is $\pi/4$.
(You can do both parts with the same calculation.)

Solution. The goal is to show that the limit $\displaystyle\lim_{N\to\infty}\int_1^N \frac{1}{1+x^2}\,dx$ exists and has the value $\pi/4$. Now

$$\int_1^N \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_1^N = \arctan(N) - \arctan(1).$$

When N gets large, the function $\arctan(N)$ approaches a horizontal asymptote at height $\pi/2$; that is, $\displaystyle\lim_{N\to\infty}\arctan(N) = \pi/2$. Therefore

$$\lim_{N\to\infty}\int_1^N \frac{1}{1+x^2}\,dx \qquad \text{exists,}$$

and the value of the limit is

$$\frac{\pi}{2} - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Thus the improper integral $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ does converge and has the indicated value.

2. By solving the initial-value problem

$$y' + \frac{y}{x} = 3x, \qquad y(1) = 5,$$

show that $y(2) = 6$.
Hint: use a suitable integrating factor.

Solution. The integrating factor is $\exp(\int \frac{1}{x}\,dx) = e^{\ln(x)} = x$. Multiplying the equation by the integrating factor gives the equivalent equation $xy' + y = 3x^2$, or $(xy)' = 3x^2$. Integrating gives $xy = x^3 + C$ for some constant C. Invoking the initial condition gives $5 = 1 + C$, so $C = 4$. Therefore the solution of the differential equation is $xy = x^3 + 4$, or equivalently $y = x^2 + \frac{4}{x}$. Consequently, $y(2) = 2^2 + \frac{4}{2} = 6$.

3. The surface area of a hemisphere of radius 1 is equal to 2π. Verify this fact by computing the area of the surface obtained by revolving the curve
$$x = \cos(t), \qquad y = \sin(t), \qquad 0 \le t \le \pi/2$$
about the y-axis. (The curve is one quarter of a circle.)

Solution. The arc length element in this problem is
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{\sin^2 t + \cos^2 t}\,dt = dt.$$
The surface area is therefore
$$\int_*^{**} 2\pi x\,ds = \int_0^{\pi/2} 2\pi\cos(t)\,dt = 2\pi\sin(t)\Big|_0^{\pi/2} = 2\pi(1-0) = 2\pi.$$

4. Suppose f is an increasing function whose graph is concave down. If $f(2) = 1.1$, $f(3) = 1.25$, and $f(4) = 1.33$, find an approximate value for the integral $\int_2^4 f(x)\,dx$ with error less than 4%. Explain how you know that the error is less than 4%.
Hint: for a curve of the indicated type, the trapezoidal approximation is an under-estimate of the area, and the midpoint approximation is an over-estimate of the area.

Solution. The question can be answered by using the simplest possible trapezoidal and midpoint approximations, without even subdividing the interval $[2, 4]$.

The area of a trapezoid whose base is the interval $[2, 4]$ and whose upper side joins the endpoints of the curve is (base) $\times$ (average altitude) $= 2 \times \frac{1}{2}(1.1 + 1.33) = 2.43$. This value is a lower estimate for the area under the curve (because the function is concave down). The area of a rectangle whose base is the interval $[2, 4]$ and whose height is the value of the function at the midpoint is $2 \times 1.25 = 2.50$. This value is an upper estimate for the area under the curve (because the function is concave down). Thus
$$2.43 \le \int_2^4 f(x)\,dx \le 2.50.$$

Therefore both the trapezoidal approximation and the midpoint approximation differ from the true value of the integral by no more than $2.50 - 2.43 = 0.07$. Since $0.07/2.43 \approx 0.0288$, both the trapezoidal approximation and the midpoint approximation are accurate not only within 4% but even within 3%.

The preceding discussion suffices to answer the question, but one can do better by using the trapezoidal approximation with two subintervals $[2, 3]$ and $[3, 4]$, each of width 1. That improved approximation equals $\frac{1}{2}(1.1 + 1.25) + \frac{1}{2}(1.25 + 1.33) = 2.465$. The midpoint approximation with two subintervals would require the values $f(2.5)$ and $f(3.5)$, which are not given. Using the midpoint approximation previously computed and the second trapezoidal approximation shows that

$$2.465 \le \int_2^4 f(x)\,dx \le 2.50.$$

Therefore both this second trapezoidal approximation and the midpoint approximation differ from the true value of the integral by no more than $2.50 - 2.465 = 0.035$. Since $0.035/2.465 \approx 0.014$, both the second trapezoidal approximation and the midpoint approximation have an error less than 2%.

A further improvement is to average the second trapezoidal approximation and the midpoint approximation to get $\frac{1}{2}(2.465 + 2.50) = 2.4825$. This value differs from the true value of the integral by at most $\frac{1}{2}(2.50 - 2.465) = 0.0175$. Hence the relative error is at most $0.0175/2.465 \approx 0.007$, or less than 1%.

5. Do **either** part (a) **or** part (b), whichever you prefer.

   (a) Show that $\lim_{n\to\infty} (-1)^n \sin(1/n) = 0$.

   Solution. A popular inadequate answer is:

   > "Since $\lim_{n\to\infty} \sin(1/n) = \sin(0) = 0$, and anything times 0 is 0, then $\lim_{n\to\infty}(-1)^n \sin(1/n) = 0$." WRONG!

   Although it is true that $\lim_{n\to\infty} \sin(1/n) = 0$ (because $\sin(x)$ is a continuous function at $x = 0$), the argument is erroneous because the limit $\lim_{n\to\infty}(-1)^n$ does not exist.

One way to overcome the difficulty is to apply the squeeze theorem by setting $a_n = -\sin(1/n)$ and $b_n = (-1)^n \sin(1/n)$ and $c_n = \sin(1/n)$. Then $a_n \le b_n \le c_n$ for every n, and $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} c_n$. Therefore $\lim_{n\to\infty} b_n = 0$ by the squeeze theorem.

(b) Kim says that the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 4^n}{5^n}$ diverges by the rule for geometric series because $\dfrac{2+4}{5} > 1$. Lee says that the series converges to the value $\dfrac{20}{3}$.
Explain who (if anyone) is right, and why.

Solution. Kim is wrong because $2^n + 4^n \ne (2+4)^n$. (Check the case $n = 2$, for example.) In fact, since $2^n < 4^n$, the numerator of the fraction is less than $4^n + 4^n$ or 2×4^n. Therefore

$$\sum_{n=0}^{\infty} \frac{2^n + 4^n}{5^n} < \sum_{n=0}^{\infty} 2 \times \frac{4^n}{5^n} = 2 \times \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n,$$

so the original series is smaller than twice a convergent geometric series. Thus the original series does converge.
Moreover, by the linearity property of series (see Theorem 8 on page 591 of the textbook),

$$\sum_{n=0}^{\infty} \frac{2^n + 4^n}{5^n} = \sum_{n=0}^{\infty} \frac{2^n}{5^n} + \sum_{n=0}^{\infty} \frac{4^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n.$$

The original series is thus the sum of two geometric series, both of which converge since the ratios $2/5$ and $4/5$ are both positive numbers less than 1. By the formula for the sum of a geometric series,

$$\sum_{n=0}^{\infty} \frac{2^n + 4^n}{5^n} = \frac{1}{1 - \frac{2}{5}} + \frac{1}{1 - \frac{4}{5}} = \frac{5}{3} + 5 = \frac{20}{3}.$$

Therefore Lee is correct.