Calculus

Math 172-502 Third Examination Fall 2005

Work all five problems. These are essay questions. To obtain maximal credit, show your work and explain your reasoning.

1. State the following convergence tests for infinite series.

   (a) the ratio test

   (b) the alternating series test

   Solution. See the textbook, pages 609 (ratio test) and 605 (alternating series test).

2. Show that the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ \ln(n)}$ diverges.

   Hint: use the integral test and the substitution $u = \ln(x)$.

   Solution. Apply the integral test with $f(x)$ equal to the function $\frac{1}{x\ \ln(x)}$ (which is positive and decreasing when $x \geq 3$). The substitution $u = \ln(x)$ shows that $\int \frac{1}{x\ \ln(x)}\, dx = \ln(\ln(x))$, so

$$\int_3^{\infty} \frac{1}{x\ \ln(x)}\, dx = \lim_{N\to\infty} [\ln(\ln(N)) - \ln(\ln(3))] = \infty.$$

   The integral diverges, so the series $\sum_{n=3}^{\infty} \frac{1}{n\ \ln(n)}$ diverges too.

   (This is problem 11, page 604, from the suggested homework problems.)

3. Show that the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}\, x^n$ converges when $-\frac{1}{2} \leq x < \frac{1}{2}$ and diverges for all other values of x.
   (Remember that you usually need to apply one method to find the radius of convergence and a different method to test the endpoints of the interval.)

Solution. Since $\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}x^{n+1}}{n+1}}{\frac{2^n x^n}{n}}\right| = 2|x|\lim_{n\to\infty}\frac{n}{n+1} = 2|x|$, the ratio test implies that the series converges when $|x| < \frac{1}{2}$ and diverges when $|x| > \frac{1}{2}$.

At the right-hand endpoint where $x = \frac{1}{2}$, the series equals $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges (it is the harmonic series). At the left-hand endpoint where $x = -\frac{1}{2}$, the series equals $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges by the alternating series test.

Thus the interval of convergence of the power series is $[-\frac{1}{2}, \frac{1}{2})$.

4. Use an appropriate test to show that the series $\displaystyle\sum_{n=1}^{\infty}\frac{n+n^2}{1+n^4}$ converges.

Solution. The idea is that when n is large, the fraction $\dfrac{n+n^2}{1+n^4}$ is nearly equal to $\dfrac{n^2}{n^4}$ or $\dfrac{1}{n^2}$, and $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ converges. Here are two ways to make this idea precise.

Inequality comparison: Since $\dfrac{n+n^2}{1+n^4} < \dfrac{n+n^2}{n^4} = \dfrac{1}{n^3} + \dfrac{1}{n^2}$, and the series $\sum_{n=1}^{\infty}\dfrac{1}{n^3}$ and $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ are both convergent p-series ($p > 1$), our series $\sum_{n=1}^{\infty}\dfrac{n+n^2}{1+n^4}$ converges because it is smaller than the sum of two convergent series.

Limit comparison: Since

$$\lim_{n\to\infty}\frac{n+n^2}{1+n^4}\bigg/\frac{1}{n^2} = \lim_{n\to\infty}\frac{\frac{1}{n}+1}{\frac{1}{n^4}+1} = 1,$$

and $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ is a convergent p-series, our series $\sum_{n=1}^{\infty}\dfrac{n+n^2}{1+n^4}$ converges by the limit comparison test.

5. Do **either** part (a) **or** part (b), whichever you prefer.

(a) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,2^n} = \ln(2)$.

Hint: look at what happens if you integrate the geometric series $1 + x + x^2 + x^3 + \cdots$.

Solution. Since $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ when $|x| < 1$, integrating gives

$$-\ln(1-x) = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad \text{when } |x| < 1.$$

(In principle there is a constant of integration, but setting $x = 0$ shows that the integration constant equals 0.) Substitute $x = \frac{1}{2}$ in this equation to obtain

$$-\ln\left(1 - \frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(1/2)^n}{n}.$$

The right-hand side equals our series $\sum\limits_{n=1}^{\infty} \frac{1}{n\,2^n}$, and the left-hand side simplifies to $\ln(2)$.

(b) Show that $\displaystyle\sum_{n=0}^{\infty} (n+1)\left(\tfrac{3}{4}\right)^n = 16$.

Hint: look at what happens if you differentiate the geometric series $1 + x + x^2 + x^3 + \cdots$.

Solution. Since $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ when $|x| < 1$, differentiating gives

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n \qquad \text{when } |x| < 1.$$

Substitute $x = \frac{3}{4}$ in this equation and observe that $\dfrac{1}{(1-\frac{3}{4})^2} = 16$.