

Calculus

Math 172-502

Final Examination

Fall 2005

Work all seven problems. These are essay questions. To obtain maximal credit, show your work and explain your reasoning.

1. Show that the curve described by the equation

$$3x = 2(y - 1)^{3/2}, \quad 1 \leq y \leq 16,$$

has length equal to 42.

Solution. (This problem is based on exercise 13 on page 570 of the textbook.) Since $x = (2/3)(y-1)^{3/2}$, it is easy to compute that $dx/dy = (y-1)^{1/2}$. Therefore the arclength element

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{y} dy.$$

The length of the curve is obtained by integrating ds :

$$\int_1^{16} \sqrt{y} dy = \frac{2}{3} y^{3/2} \Big|_1^{16} = \frac{2}{3} (64 - 1) = 42.$$

2. An aquarium 2 meters long, 1 meter wide, and 1 meter deep is full of water. Show that the work needed to pump half of the water out of the aquarium is 2,450 joules. (The density of water is 1,000 kg/m³, and the acceleration of gravity is 9.8 m/sec².)

Solution. (This is exercise 15 on page 448 of the textbook.) Let y denote the distance measured down from the top of the tank. The volume of a thin slab of water with thickness dy at depth y equals $2 \times 1 \times dy$. Multiplying this volume by $9.8 \times 1,000$ gives a weight (that is, a force). The work done to lift this slab of water to the top of the tank (a distance y) is $2 \times 9.8 \times 1,000 \times y dy$. The total work is

$$\int_0^{1/2} 2 \times 9.8 \times 1,000 \times y dy = 9.8 \times 1,000 \times y^2 \Big|_0^{1/2} = 2,450.$$

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3. By solving the differential equation $\frac{dy}{dx} + y = \frac{\sqrt{x}}{e^x}$ with initial condition $y(0) = 3$, show that $y(9) = 21/e^9$.

Solution. (This problem is based on exercise 11 on page 570 of the textbook.) Multiply the equation by the integrating factor e^x to get

$$e^x \frac{dy}{dx} + e^x y = \sqrt{x} \quad \text{or} \quad (e^x y)' = \sqrt{x}.$$

Integrate to get $e^x y = \frac{2}{3}x^{3/2} + C$. Evaluate the integration constant C by using the initial condition: when $x = 0$, $y = 3$, so $3e^0 = 0 + C$ or $C = 3$. Then set $x = 9$ to get

$$e^9 y(9) = \frac{2}{3}9^{3/2} + 3 = 21.$$

Hence $y(9) = 21/e^9$.

4. Show that the improper integral $\int_0^{\pi/4} \frac{\sec^2(\theta)}{1 - \tan(\theta)} d\theta$ diverges.

Solution. (This problem is based on exercise 16 on page 519 in the textbook.) The integral is improper because the denominator equals 0 when $\theta = \pi/4$. Substituting $u = 1 - \tan(\theta)$ and $du = -\sec^2(\theta) d\theta$ shows that

$$\int \frac{\sec^2(\theta)}{1 - \tan(\theta)} d\theta = -\ln(1 - \tan(\theta)) = \ln\left(\frac{1}{1 - \tan(\theta)}\right).$$

The improper integral is evaluated as the limit

$$\lim_{b \rightarrow \pi/4} \left[\ln\left(\frac{1}{1 - \tan(\theta)}\right) \right]_0^b = \lim_{b \rightarrow \pi/4} \ln\left(\frac{1}{1 - \tan(b)}\right) - 0.$$

When b approaches $\pi/4$ (from below), $1 - \tan(b)$ approaches 0 (from the positive side), so the reciprocal $1/(1 - \tan(b))$ grows without bound, and so does the logarithm $\ln(1/(1 - \tan(b)))$. In other words, the limit does not exist, and that is what it means for the integral to diverge.

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5. Use substitution and/or integration by parts to show that

$$\int_0^1 \arcsin(x) dx = \frac{\pi}{2} - 1.$$

Solution. (This problem is based on exercise 36 on page 519 of the textbook.) If $u = \arcsin(x)$, then $\sin(u) = x$, so $dx = \cos(u) du$. Since $u = 0$ when $x = 0$ and $u = \pi/2$ when $x = 1$, substituting in the integral gives the new integral $\int_0^{\pi/2} u \cos(u) du$. Integrating by parts gives

$$u \sin(u) \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(u) du = \left[u \sin(u) + \cos(u) \right]_0^{\pi/2} = \frac{\pi}{2} - 1.$$

Another way to solve the problem is to integrate by parts first, getting

$$x \arcsin(x) \Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \frac{\pi}{2} + \int_0^1 \frac{-x}{\sqrt{1-x^2}} dx.$$

In the remaining integral, substitute $u = 1 - x^2$ and $du = -2x dx$ to get

$$\frac{\pi}{2} + \int_1^0 \frac{\frac{1}{2} du}{\sqrt{u}} = \frac{\pi}{2} + \left[\sqrt{u} \right]_1^0 = \frac{\pi}{2} - 1.$$

6. Explain why the approximation

$$\int_0^{0.1} \frac{1}{1+x^9} dx \approx 0.09999999999000000000$$

is correct to 20 decimal places.

Solution. (This problem is similar to Example 8 on pages 621–622 of the textbook.) When $|x| < 1$, one has the geometric series expansion

$$\frac{1}{1+x^9} = 1 - x^9 + x^{18} - x^{27} + \cdots,$$

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so integrating term-by-term gives

$$\int_0^{0.1} \frac{1}{1+x^9} dx = \left[x - \frac{1}{10}x^{10} + \frac{1}{19}x^{19} - \dots \right]_0^{0.1}.$$

Evaluating at the limits of integration gives

$$\int_0^{0.1} \frac{1}{1+x^9} dx = 0.1 - 10^{-11} + \frac{1}{19} \times 10^{-19} - \dots.$$

The given decimal is the sum of the first two terms on the right-hand side. By the alternating series remainder estimate, the sum of these first two terms is smaller than the value of the integral (since the next term is positive) and differs from the value of the integral by less than $\frac{1}{19} \times 10^{-19}$, which is about 5×10^{-21} . Consequently, the remainder of the series (after the first two terms) adds something in the twenty-first decimal place, and this does not change the first twenty decimal places of the answer.

7. Show that the infinite series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

For extra credit (5 points), show that the value of this sum is 6.

Solution. (This problem is based on exercise 28 on page 647 of the textbook.) The ratio test applies. Since

$$\lim_{n \rightarrow \infty} \frac{(n+1)^2}{2^{n+1}} \bigg/ \frac{n^2}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 + \frac{1}{n} \right)^2 = \frac{1}{2} < 1,$$

the series converges.

To solve the extra credit part, start from the geometric series formula

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Differentiate to get

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

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Multiply by x to get

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Differentiate again to get

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{(1-x)^2 \times 1 - x \times 2 \times (1-x) \times (-1)}{(1-x)^4} = \frac{1+x}{(1-x)^3}.$$

Multiply by x to get

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}.$$

Substitute $x = 1/2$ to get

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2}(1 + \frac{1}{2})}{(1 - \frac{1}{2})^3} = 6.$$