Notes on the Zermelo-Fraenkel axioms for set theory

Russell's paradox shows that one cannot talk about "the set of all sets" without running into a contradiction. In order to have a self-consistent language for talking about sets, one needs some rules that say what sets exist and what sentences are legitimate descriptions of sets.

The most commonly used system of axioms for set theory is called "ZFC" in honor of Ernst Friedrich Ferdinand Zermelo (1871–1953) and Adolf Abraham Halevi Fraenkel (1891–1965). (The letter "C" refers to the Axiom of Choice discussed below.) Although there is no universal agreement on the order of the axioms, the exact wording of the axioms, or even how many axioms there are, most mathematicians will accept the following list.

1 Axiom of extension

Two sets are equal if and only if they have the same elements: in symbols,

$$\forall A \, \forall B \, ((A = B) \iff \forall x \, ((x \in A) \iff (x \in B))).$$

Although Axiom 1 describes when two sets are equal, the axiom does not guarantee that any sets exist: conceivably the whole theory could be vacuous. The next axiom fills this vacuum by stating that at least one set does exist.

2 Axiom of the empty set

There exists a set with no elements: in symbols,

$$\exists A \, \forall x \, (x \notin A).$$

By Axiom 1, the empty set (denoted $\varnothing$) is unique!

The next three axioms describe ways to build new sets from existing ones.

3 Axiom of unordered pairs

Sets $\{x, y\}$ exist: in symbols,

$$\forall x \, \forall y \, \exists A \, \forall z \, ((z \in A) \iff ((z = x) \vee (z = y))).$$

Axiom 3 also implies the existence of singleton sets: the set $\{x\}$ is equal to the unordered pair $\{x, x\}$. A standard way to represent the *ordered* pair (x, y) is the set $\{\{x\}, \{x, y\}\}$, which exists by repeated application of Axiom 3.

4 Axiom of unions

Unions exist. In the following symbolic form of the axiom, think of A as a set of sets and B as the union of those sets:

$$\forall A\,\exists B\,\forall x\,((x \in B) \iff \exists c\,((c \in A) \wedge (x \in c))).$$

Finite sets like $\{x, y, z\}$ can be constructed by Axioms 3 and 4.

5 Axiom of the power set

Power sets exist. In symbols,

$$\forall A\,\exists B\,((x \in B) \iff (x \subseteq A)).$$

Here the statement "$x \subseteq A$" is a shorthand expression for the statement "$\forall y\,((y \in x) \implies (y \in A))$".

6 Axiom of infinity

An infinite set exists. One way to write this statement in symbols is:

$$\exists A\,((\varnothing \in A) \wedge \forall x\,((x \in A) \implies ((x \cup \{x\}) \in A))).$$

An infinite set of the indicated form contains a copy of the natural numbers, modeled as follows: first 0 corresponds to the empty set $\varnothing$, then 1 corresponds to $\{\varnothing\}$, then 2 corresponds to $\{\varnothing, \{\varnothing\}\}$, and so on. One would like to conclude that the set of natural numbers exists, since there is a rule for identifying the natural numbers as a subset of a previously constructed set.

To justify this conclusion, one has to know that a rule for selecting a subset necessarily defines a set. Hence the next axiom is needed.

7 Axiom of selection

If P is an open sentence, and A is a set, then the expression $\{\, x \in A \mid P(x) \,\}$ defines a set: the subset of elements of A for which the property P holds.

A key point is that the set being defined is required to be a subset of some previously given set A. This requirement rules out $\{\, x \mid x \notin x \,\}$ (Russell's paradoxical "set").

Axiom 7 is actually an "axiom schema" representing an infinite collection of axioms, one for each statement P. The next axiom too is an axiom schema.

8 Axiom of replacement

The image of a set under a function is again a set. In other words, if A and B are sets, and $f\colon A \to B$ is a function with domain A and codomain B, then the image $f(A)$ is a set. (A function f may be described in set-theoretic terms as the set of ordered pairs $\{\, (a, b) \in A \times B \mid f(a) = b \,\}$.)

9 Axiom of regularity

Every non-empty set has an element that is disjoint from the set: in symbols,

$$\forall A\,((A \neq \varnothing) \implies \exists x\,((x \in A) \wedge (x \cap A = \varnothing))).$$

Another name for this axiom is "the axiom of foundation".

In contrast to most of the other axioms, Axiom 9 does not guarantee the existence of any sets. Instead the axiom rules out the existence of certain pathological sets. In particular, Axiom 9 implies that no set can be an element of itself (this is one of the exercises below).

10 Axiom of Choice

Given any infinite collection of non-empty sets, it is possible to choose (simultaneously) one element from each set. More precisely, if f is a function whose domain is a non-empty set A and whose codomain is a set B whose elements are non-empty sets, then there is a "choice function" g with the property that $g(x) \in f(x)$ for each x in A.

There are many equivalent formulations of the Axiom of Choice. One of them is that the Cartesian product of (an infinite number of) non-empty sets exists and is a non-empty set.

In 1940, Kurt Gödel proved that the Axiom of Choice is consistent with Axioms 1–9 (assuming that those axioms themselves are self-consistent). On the other hand, in 1963 Paul J. Cohen showed that the negation of the Axiom of Choice is consistent with Axioms 1–9 (again assuming that those axioms themselves are self-consistent). In other words, the Axiom of Choice is *independent* of Axioms 1–9.

Thus, just as one can do geometry either with the Parallel Postulate or without the Parallel Postulate, one can do set theory either with the Axiom of Choice or without the Axiom of Choice. Most mathematicians accept the Axiom of Choice because that axiom appears to be a natural and useful property.

On the other hand, the Axiom of Choice does have some surprising and counter-intuitive consequences. For example, the Axiom of Choice implies that every non-empty set admits a well-ordering. Also, the Axiom of Choice implies the Banach-Tarski paradox. Consequently, a few mathematicians prefer to work in the system "ZF" consisting of Axioms 1–9 without the Axiom of Choice.

For further reading

- N. Ya. Vilenkin, *Stories about Sets*, Academic Press, 1968; QA 248 V5.13. Suitable for bed-time reading, this little book is directed to "anybody, beginning with high school juniors and seniors" (according to the Foreword).
- Paul R. Halmos, *Naive Set Theory*, Van Nostrand, 1960; QA 248 H2.6. This slim volume is a very readable presentation of the elements of set theory by a master of mathematical exposition.

Exercises

(a) Use Axiom 9 (the axiom of regularity) to prove that there is no set A for which $A \in A$.
[Hint: if there were such a set A, what would Axiom 9 say about the singleton set $\{A\}$?]

(b) The description of Axiom 8 above uses the notion of the Cartesian product of two sets A and B. Use Axioms 1–7 to prove that the Cartesian product $A \times B$ does exist as a set.
[Hint: the key point is to show that a suitable set exists from which $A \times B$ can be selected.]