Examination 2

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

Notation: The symbol $\mathbb{Z}$ denotes $\{ \dots, -1, 0, 1, 2, \dots \}$ (the set of all integers), and the symbol $\mathbb{N}$ denotes $\{ 1, 2, 3, \dots \}$ (the set of positive integers).

1. Fill in the blank with the appropriate word: A relation R on a set A is called a partial ordering if the relation R is reflexive, ____________ , and antisymmetric.

   Solution. A relation R on a set A is called a partial ordering if the relation R is reflexive, transitive, and antisymmetric.

2. Suppose $f : \mathbb{Z} \to \mathbb{N}$ is defined by the property that $f(n) = n^2 + 1$ for each integer n. Let the symbol $\mathbb{O}$ denote the set $\{ 1, 3, 5, \dots \}$ (the odd positive integers). Determine $f^{-1}(\mathbb{O})$ (that is, the inverse image of the set $\mathbb{O}$).

   Solution. Notice that the symbol f^{-1} does not mean inverse function here. Indeed, the function f is not injective and so is not invertible. What the problem is asking is to find the integer values of n for which $n^2 + 1$ is an odd positive integer.

   Consider two cases for the input integer n. If n is odd, then n^2 is odd, so $n^2 + 1$ is even. Thus no odd integer belongs to the set $f^{-1}(\mathbb{O})$. On the other hand, if n is even, then n^2 is even, so $n^2 + 1$ is odd. Thus every even integer belongs to the set $f^{-1}(\mathbb{O})$. In summary, the set $f^{-1}(\mathbb{O})$ is the set of all even integers (positive, negative, and zero).

3. Let R be the relation defined on $\mathbb{Z}$ by saying that a is related to b (in symbols, aRb) when the difference $a - b$ is an odd integer. Is this relation an equivalence relation? Explain why or why not.

   Solution. An equivalence relation is supposed to be reflexive, symmetric, and transitive. If any one of these three properties fails, then the relation is not an equivalence relation.

   Reflexivity fails. No element a is related to itself, for the difference $a - a$ equals 0, which is not an odd integer. Indeed, the number 0 is an even integer (a multiple of 2), since $0 = 0 \times 2$. Thus the relation is not an equivalence relation.

   Transitivity fails too. Indeed, if aRb and bRc, then $a - b$ is odd and $b - c$ is odd. The sum of these two odd integers equals $a - c$, which thus is an even integer. Accordingly, if aRb and bRc, then it is not the case that aRc. The failure of transitivity is an alternative proof that the relation is not an equivalence relation.

   On the other hand, symmetry does hold. If $a - b$ is odd, then so is $b - a$, since the negative of an odd integer is odd. Thus aRb does imply bRa.

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4. Suppose $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are defined by the properties that $f(n) = n^2$ and $g(n) = 2^n$ for each positive integer n. Let h_1 denote the composition $f \circ g$, and let h_2 denote the composition $g \circ f$. Which of the values $h_1(4)$ and $h_2(3)$ is the larger? Explain how you know.

   Solution. The definition of function composition implies that $h_1(4) = f(g(4)) = f(2^4) = (2^4)^2 = 2^8$, and $h_2(3) = g(f(3)) = g(3^2) = 2^9$. Therefore $h_2(3)$ is larger than $h_1(4)$.

5. Consider the binary operation $*$ on $\mathbb{Z}$ defined as follows: $m * n = m + n - mn$ when m and n are integers. Is the operation $*$ associative? Explain why or why not.

   Solution. The question is whether $k * (m * n) = (k * m) * n$ for all integers k, m, and n. Since the binary operation says to add the two numbers and subtract their product,

$$\begin{aligned} k * (m * n) &= k * (m + n - mn) \\ &= k + (m + n - mn) - k(m + n - mn) \\ &= k + m + n - mn - km - kn + kmn. \end{aligned}$$

   On the other hand,

$$\begin{aligned} (k * m) * n &= (k + m - km) * n \\ &= (k + m - km) + n - (k + m - km)n \\ &= k + m - km + n - kn - mn + kmn. \end{aligned}$$

   The two expressions are equal: both are the sum of the three numbers plus the product of the three numbers minus the products of pairs of numbers. Therefore the binary operation is associative.

   Remark. There is a sneaky way to avoid making the second computation. Since ordinary addition is commutative ($m + n = n + m$), and ordinary multiplication is commutative too ($mn = nm$), and $*$ is built from these two operations, the operation $*$ evidently is commutative as well ($m * n = n * m$). Moreover, the computed expression for $k * (m * n)$ evidently is unchanged by permuting the letters k, m, and n. Accordingly,

$$\begin{aligned} k * (m * n) &= n * (k * m) && \text{(by permuting)} \\ &= (k * m) * n && \text{(by commutativity),} \end{aligned}$$

   and the proof of associativity is complete.

6. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ defined as follows:

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even,} \\ \dfrac{1-n}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Show that f is a bijection.

Solution. What needs to be shown is that f is both injective and surjective (both one-to-one and onto). Consider the two properties separately, as follows.

Injectivity. What needs to be shown is that if $f(m) = f(n)$, then $m = n$. Consider three cases.

If both m and n are even, then the definition of f implies that $m/2 = f(m) = f(n) = n/2$, and multiplying by 2 shows that $m = n$. Thus the required conclusion holds in this case.

If both m and n are odd, then the definition of f implies that $(1 - m)/2 = f(m) = f(n) = (1 - n)/2$. Multiplying by 2 yields that $1 - m = 1 - n$, subtracting 1 shows that $-m = -n$, and multiplying by -1 demonstrates that $m = n$. Thus the required conclusion holds in this case too.

Finally, suppose that the numbers m and n have opposite parity (one of the numbers is even and the other number is odd). The definition of f implies that one of the numbers $f(m)$ and $f(n)$ is positive and the other is negative or 0. This deduction contradicts the hypothesis that $f(m) = f(n)$, so this case never occurs.

In summary, whatever the values of m and n, if $f(m) = f(n)$ then $m = n$. Therefore the function f is injective.

Surjectivity. What needs to be shown is that every integer n (positive, negative, or zero) is in the image of f.

If n is positive, then $2n$ is a positive even integer, so the definition of f implies that $f(2n) = n$. Thus every positive integer is in the image of f.

The integer 0 is in the image of f because $f(1) = 0$.

Finally, suppose n is a negative integer. Then $-2n$ is a positive even integer, and $1-2n$ is a positive odd integer, so the definition of f implies that $f(1-2n) = \dfrac{1-(1-2n)}{2} = \dfrac{2n}{2} = n$. Thus every negative integer is in the image of f.

The three preceding cases show that every integer (positive, zero, or negative) is an element of the image of f: in other words, the function f is surjective.

Remark. This problem shows explicitly that the set of all integers can be put into one-to-one correspondence with a proper subset (namely, the set of positive integers). The property of being in one-to-one correspondence with a proper subset cannot happen for a finite set, but this property holds for every infinite set.

7. When $f : A \to A$ is a function whose domain and codomain are the same set A, there is an associated relation R_f on A defined by saying that a is related to b (in symbols, $aR_f b$) when $b = f(a)$. Show that symmetry of this relation R_f means that the function f is invertible and equal to its inverse function (in symbols, $f = f^{-1}$).

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Solution. Suppose that R_f is symmetric, and let a be an arbitrary element of the set A. Let b denote $f(a)$. The definition of R_f means that aR_fb, so symmetry implies that bR_fa; in other words, $a = f(b) = f(f(a))$.

The property that $a = f(f(a))$ for every a implies that f is a bijective function. Indeed, the function f is surjective, because an arbitrary element a is the image under f of the element $f(a)$. And f is injective because if $f(a_1) = f(a_2)$, then $f(f(a_1)) = f(f(a_2))$, whence $a_1 = a_2$. Being bijective, the function f is invertible.

Now $f(f(a))$ is the value at a of the composite function $f \circ f$. Since $a = f(f(a))$ for every a, the composite function $f \circ f$ is the identity function. But this property is precisely what it means to say that $f^{-1} = f$.

The preceding argument shows that if the relation R_f is symmetric, then f is invertible, and $f^{-1} = f$. The argument is reversible. If f is known to be invertible, and if $f^{-1} = f$, then the composite function $f \circ f$ must be the identity function. Accordingly, if $b = f(a)$, then $f(b) = f(f(a)) = a$. Thus $aR_fb \implies bR_fa$. In other words, if $f^{-1} = f$, then the relation R_f is symmetric.