Final Examination

Instructions. Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Consider the statement: "Every Aggie wears maroon," or (equivalently) "If x is an Aggie, then x wears maroon." Write

   a) the converse of the statement;

   b) the contrapositive of the statement.

   Solution.

   a) The converse is, "If x wears maroon, then x is an Aggie," or, in ordinary English, "Everyone who wears maroon is an Aggie."

   b) The contrapositive is, "If x does not wear maroon, then x is not an Aggie," or, in English, "Everyone who does not wear maroon is not an Aggie."

2. Does the operation of taking the Cartesian product of sets distribute over the operation of taking the intersection of sets? In symbols, can you say that

   $$A \times (B \cap C) = (A \times B) \cap (A \times C) \qquad \text{for all sets } A,\ B, \text{ and } C\,?$$

   Explain why or why not.

   Solution. The indicated distributive property does hold.

   Indeed, a general element of the set $A \times (B \cap C)$ has the form (a, d), where $a \in A$ and $d \in B \cap C$. In particular, $d \in B$, so the ordered pair (a, d) belongs to the set $A \times B$; and since $d \in C$, the ordered pair (a, d) belongs to the set $A \times C$ too. By the definition of intersection, the ordered pair (a, d) belongs to the set $(A \times B) \cap (A \times C)$.

   In the other direction, consider a general element (a, d) of the set $(A \times B) \cap (A \times C)$. Then $(a, d) \in A \times B$, so $a \in A$ and $d \in B$. Also $(a, d) \in A \times C$, so $d \in C$. By the definition of intersection, $d \in B \cap C$. Thus (a, d) is an element of the set $A \times (B \cap C)$.

   The preceding two paragraphs show that $A \times (B \cap C)$ is a subset of $(A \times B) \cap (A \times C)$, and $(A \times B) \cap (A \times C)$ is a subset of $A \times (B \cap C)$. Therefore the two sets are equal.

3. Suppose a function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as follows: $f(n) = 3n + 1$ for each integer n.

   a) Is the function f injective?

   b) Is the function f surjective?

   Explain why or why not.

   Solution.

a) The function f is injective. Indeed, if $f(n) = f(m)$, then $3n + 1 = 3m + 1$. Subtracting 1 from both sides shows that $3n = 3m$, and dividing by 3 shows that $n = m$. Thus $(f(n) = f(m)) \implies (n = m)$, which is precisely the meaning of injectivity.

b) The function f is not surjective. Indeed, the integer 2 is not an element of the image, for if $3n + 1 = 2$, then $3n = 1$, but there is no integer n for which $3n = 1$.

4. Suppose a relation is defined on subsets of the integers by saying that set A is related to set B when A is a subset of B (in symbols, $A \subseteq B$).

   a) Is this relation reflexive?

   b) Is this relation symmetric?

   c) Is this relation transitive?

   Explain why or why not.

   Solution.

   a) The relation is reflexive, for every set is a subset of itself.

   b) The relation is not symmetric. Indeed, the set $\{1\}$ is a subset of $\{1, 2\}$, but the set $\{1, 2\}$ is not a subset of $\{1\}$.

   c) The relation is transitive: if A is a subset of B, and B is a subset of C, then A is a subset of C.

5. Here are five concepts from the course that start with the letter p:

   a) partial ordering on a set

   b) partition of a set

   c) permutation of a set

   d) pigeonhole principle

   e) power set of a set

   Explain the meaning of **three** of these concepts.

   Solution. You can find the definitions of these concepts by consulting the index of the textbook.

6. Use the method of mathematical induction to prove *Bernoulli's inequality*: namely, if x is a real number greater than -1, and n is a positive integer, then $(1 + x)^n \geq 1 + nx$. (Treat the number x as a fixed quantity and the number n as the induction variable.)

   Solution. Evidently $(1 + x)^1 \geq 1 + 1 \cdot x$ (actually equality holds), so the basis step is valid.

Suppose now that $(1+x)^k \geq 1+kx$ for some positive integer k. Since $x > -1$, the quantity $1+x$ is positive. Multiplying the induction hypothesis by this positive quantity $1+x$ implies that

$$(1+x)^{k+1} \geq (1+x)(1+kx) = 1 + x + kx + kx^2 \geq 1 + (k+1)x$$

since $kx^2 \geq 0$. Thus the inequality holds for integer $k+1$ if the inequality holds for integer k.

Since the basis step holds, and the induction step holds, the method of mathematical induction implies that the inequality holds for every positive integer.

Bonus problem. The definition of "f is continuous at c" that you will learn in Math 409 says: "For every positive ε, there exists a positive δ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$." Write the negation of this statement without using the word "not."
Hint: There is an implicit universal quantifier hidden in the word "whenever."

Solution. The original statement can be symbolized as follows:

$$\forall \varepsilon \ \exists \delta \ \forall x \ (|x - c| < \delta) \implies (|f(x) - f(c)| < \varepsilon).$$

The rules for negating quantifiers and negating implications imply that the negation of the statement is the following:

$$\exists \varepsilon \ \forall \delta \ \exists x \ (|x - c| < \delta) \wedge (|f(x) - f(c)| \geq \varepsilon),$$

or in words, there exists a positive ε such that for every positive δ there exists an x for which $|x - c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon$.