Several Variable Calculus

1. If the boundary of a square in the xz-plane is traversed from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 0, 1)$ to $(0, 0, 1)$ to $(0, 0, 0)$, what is the compatible direction for the vector normal to the square?

   Solution. The direction of the boundary is counterclockwise as seen from the negative part of the y-axis, so the compatible direction for the vector normal to the square is the $-\hat{\jmath}$ direction.

2. Find the flux of the vector field $\vec{F}(x, y, z) = z\hat{k}$ across the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane, that is, $\iint \vec{F} \cdot \hat{n}\, \mathrm{d}S$ (where the surface is oriented upward).

   Solution. The paraboloid is the level surface on which $z + x^2 + y^2 = 1$. The gradient vector $\langle 2x, 2y, 1\rangle$ is normal to the surface, and since the $\hat{k}$ component of the gradient vector is positive, this normal is oriented upward, as indicated in the statement of the problem. On the surface, $\vec{F} = (1-x^2-y^2)\hat{k}$. The dot product of this vector field with the normal vector equals $1-x^2-y^2$, so $\iint \vec{F} \cdot \hat{n}\, \mathrm{d}S = \iint_R (1 - x^2 - y^2)\, \mathrm{d}A$, where the integration region R is the disk in the xy-plane bounded by the circle $x^2 + y^2 = 1$.

   The integral is most easily computed in polar coordinates:

$$\begin{aligned}\iint_R (1 - x^2 - y^2)\, \mathrm{d}A &= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, \mathrm{d}r\, \mathrm{d}\theta \\ &= 2\pi \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 \\ &= \frac{\pi}{2}.\end{aligned}$$

3. If $\vec{G}(x, y, z) = 3x^2\hat{\imath} + 4y^3\hat{\jmath} + 5z^4\hat{k}$, find the value of the line integral $\int_C \vec{G} \cdot d\vec{r}$ when C is the boundary of the surface in problem 2 (oriented counterclockwise as seen from above).

   Solution. Here are three different ways to show that the answer is 0.

   By Stokes's theorem, the integral equals $\iint \nabla \times \vec{G} \cdot \hat{n}\, \mathrm{d}S$, where the area integral is taken over the surface in problem 2. A routine calculation shows that $\nabla \times \vec{G} = 0$, so the integral equals 0.

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Since the curve lies in the xy-plane, where the vector field $\vec{G}$ reduces to $3x^2\hat{\imath} + 4y^3\hat{\jmath}$, you could just as well apply Green's theorem:

$$\begin{aligned}\int_C \vec{G}\cdot d\vec{r} = \int_C 3x^2\,\mathrm{d}x + 4y^3\,\mathrm{d}y \\ &= \iint_R \left(\frac{\partial}{\partial x}\left(4y^3\right) - \frac{\partial}{\partial y}\left(3x^2\right)\right)\mathrm{d}A \\ &= \iint_R 0\,\mathrm{d}A \\ &= 0,\end{aligned}$$

where $\boldsymbol{R}$ is the region in the xy-plane inside the circle $x^2 + y^2 = 1$.

Alternatively, observe that $\vec{G} = \nabla(x^3 + y^4 + z^5)$, so $\int_C \vec{G}\cdot \mathrm{d}\vec{r}$ equals the difference of the values of $\vec{G}$ at the ending and starting points of the curve. But these two points are the same for a closed curve, so the integral equals 0. (In other words, the vector field $\vec{G}$ is a conservative vector field.)