Several Variable Calculus

1. Find parametric equations for the line passing through the points $(1, 2, 0)$ and $(0, 3, 4)$.

   Solution. Subtracting the coordinates of one point from the other gives a vector pointing in the direction of the line: namely, $\langle -1, 1, 4 \rangle$. The equation of the line can then be written in vector form:

   $$\vec{r}(t) = \langle 1, 2, 0 \rangle + t \langle -1, 1, 4 \rangle$$

   or in coordinate form:

   $$x(t) = 1 - t$$
   $$y(t) = 2 + t$$
   $$z(t) = 4t.$$

   There are other correct answers: you could choose a different starting point for the line, and you could multiply the direction vector by a constant. For example, $\langle 0, 3, 4 \rangle + t \langle 1, -1, -4 \rangle$ is another valid description of the line. You could even divide the direction vector by its length to make a unit direction vector, but that step is not necessary in this problem.

   The question does not ask for the symmetric form of the line. That form would arise by eliminating the parameter t to obtain:

   $$-(x - 1) = y - 2 = z/4.$$

2. Suppose a curve has the following vector equation:

   $$\vec{r}(t) = \langle 1, t, t^2 \rangle.$$

   Find the unit tangent vector $\vec{T}$ to the curve at the point where $t = 1$.

   Solution. The unit tangent vector is the velocity vector (the derivative of the position vector) divided by its length. Now $\vec{r}'(t) = \langle 0, 1, 2t \rangle$, so $\vec{r}'(1) = \langle 0, 1, 2 \rangle$, and $|\vec{r}'(1)| = \sqrt{5}$. Accordingly,

   $$\vec{T}(1) = \frac{\vec{r}'(1)}{|\vec{r}'(1)|} = \left\langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle.$$

Several Variable Calculus

3. Suppose two curves $\vec{r}_1(t)$ and $\vec{r}_2(t)$ intersect at the point $(0,0,0)$. What would your strategy be for finding the *angle* of intersection at this point?

   Carry out your strategy when $\vec{r}_1(t) = \langle t, t^2, t^2 \rangle$ and $\vec{r}_2(t) = \langle t, t, t^4 \rangle$.

   Solution. The goal is to find the angle between the two velocity vectors (or tangent vectors). The dot product of these vectors equals the product of their lengths times the cosine of the angle between them, so computing the lengths and the dot product makes it possible to solve for the angle.

   Notice that both curves are parametrized such that they pass through the point $(0,0,0)$ when $t = 0$. Now $\vec{r}_1'(t) = \langle 1, 2t, 2t \rangle$, so $\vec{r}_1'(0) = \langle 1, 0, 0 \rangle$. And $\vec{r}_2'(t) = \langle 1, 1, 4t^3 \rangle$, so $\vec{r}_2'(0) = \langle 1, 1, 0 \rangle$. Now compute the dot product two ways:

$$\langle 1,0,0 \rangle \cdot \langle 1,1,0 \rangle = 1 \times 1 + 0 \times 1 + 0 \times 0 = 1$$
$$= |\langle 1,0,0 \rangle| \, |\langle 1,1,0 \rangle| \, \cos(\theta) = \sqrt{2} \, \cos(\theta).$$

   Accordingly, $\cos(\theta) = \frac{1}{\sqrt{2}}$, so the angle θ equals $\frac{\pi}{4}$ radians (or 45°).