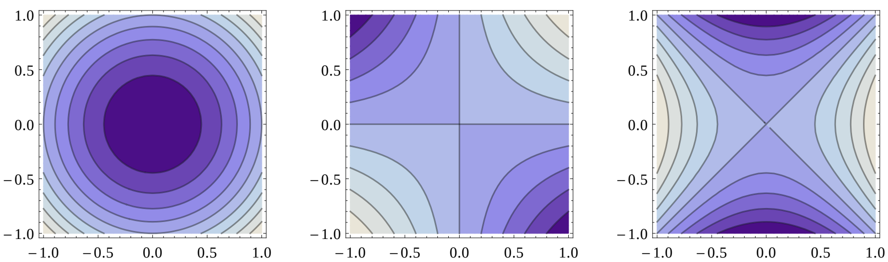
Several Variable Calculus

1. Suppose $f(x, y) = xy$. Which of the following pictures is a contour map of this function? Explain how you know.



(A) (B) (C)

Solution. Setting xy equal to a constant produces a hyperbola with the two coordinate axes as asymptotes (see page A21 in Appendix C to the textbook), so the answer is plot (B). (The special case when $xy = 0$ produces the coordinate axes themselves.)

Alternatively, you can get the answer without using any knowledge of conic sections. Consider a typical level curve, say $xy = 1$. When x gets close to 0, the value of y has to get large in magnitude (since $y = 1/x$). In picture (A), the value of y gets close to 0 when x does, and in picture (C), the value of y approaches a finite value when x gets close to 0. Therefore the answer has to be (B).

2. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$ exist? Explain why or why not.

Solution. (This problem is Exercise 12 on page 739 of the textbook.) This two-dimensional limit does not exist, for there are different limits along different paths approaching the point $(0, 0)$.

Indeed, on the x-axis (where $y = 0$), the function equals x^2/x^2, or 1, so the limit as $x \to 0$ equals 1. But along the line where $y = -x$, the function equals 0, so the limit at $(0, 0)$ along this line equals 0. And along the line where $y = x$, the function equals $(2x)^2/(2x^2)$, or 2, so the limit at $(0, 0)$ along this line equals 2. Since there are different limits at $(0, 0)$ along different paths, the two-dimensional limit does not exist.

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Another way to analyze the problem is to expand the numerator to simplify the function:

$$\frac{(x+y)^2}{x^2+y^2} = \frac{x^2+2xy+y^2}{x^2+y^2} = 1 + 2\frac{xy}{x^2+y^2}.$$

And now the problem reduces to Example 2 on page 734 of the textbook.

3. Suppose $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Compute $\dfrac{\partial f}{\partial z}(0,3,4)$.

Solution. (This problem is Exercise 14 on page 746 of the textbook.) To compute the z partial derivative, treat the variables x and y as constants. Then

$$\frac{\partial f}{\partial z}(0,3,4) = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(0+0+2z)\,\Big|_{(x,y,z)=(0,3,4)} = \frac{4}{5}.$$

Alternatively, since x and y are treated as constants, you could plug in their values at the beginning of the calculation: $f(0,3,z) = \sqrt{9+z^2}$, so

$$\frac{\partial f}{\partial z}(0,3,4) = \frac{d}{dz}\sqrt{9+z^2}\,\Big|_{z=4} = \frac{1}{2}(9+z^2)^{-1/2}(2z)\,\Big|_{z=4} = \frac{4}{5}.$$