Linear Algebra

Both problems concern the inner-product space $C[0,1]$ of continuous functions on the interval $[0,1]$ with the following inner product:

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

1. For which positive number a does the function $f(x) = ax^4$ have norm equal to 1?

   Solution. Since $\|f\|^2 = \langle f, f\rangle = \int_0^1 a^2x^8\,dx = \left[a^2\frac{x^9}{9}\right]_0^1 = a^2/9$, we want $a^2/9 = 1$, so $a = 3$.

2. Find the cosine of the angle between the functions $f(x) = \sqrt{x}$ and $g(x) = 1$.

   Solution. Since $\langle f, g\rangle = \|f\|\,\|g\|\,\cos(\theta)$, compute:

$$\langle f, g\rangle = \int_0^1 \sqrt{x}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_0^1 = \frac{2}{3},$$

$$\|f\|^2 = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}, \qquad \text{so } \|f\| = \frac{1}{\sqrt{2}},$$

$$\|g\|^2 = \int_0^1 1^2\,dx = 1, \qquad \text{so } \|g\| = 1.$$

   Therefore

$$\cos(\theta) = \frac{\langle f, g\rangle}{\|f\|\,\|g\|} = \frac{2/3}{1/\sqrt{2}} = \frac{2\sqrt{2}}{3}.$$

   [The question did not ask for the value of θ, which is $\arccos(2\sqrt{2}/3)$. Your calculator will give you an approximate value of either 0.34 radians or 19.5 degrees.]