Linear Algebra

Write your **name**: Answer Key

In **problems 1–5**, circle the correct answer. (5 points each)

1. If A is a 12×5 matrix (that is, A has 12 rows and 5 columns), then the null space of A has dimension at least 7. `True` `False`

   Solution. False. The null space is a subspace of R^5, so its dimension might be anything between 0 and 5, but the dimension cannot possibly be as large as 7. What is true is that the dimension of the null space of A^T is at least 7.

2. The function $L : R^2 \to R^1$ defined by $L(\mathbf{x}) = \|\mathbf{x}\|$ (that is, the norm of $\mathbf{x}$) is a linear transformation. `True` `False`

   Solution. False. If $\mathbf{x} \neq \mathbf{0}$, then $\|-\mathbf{x}\| \neq -\|\mathbf{x}\|$, so the function does not preserve scalar multiplication. Moreover, $\|\mathbf{x}+\mathbf{y}\|$ is not always equal to $\|\mathbf{x}\| + \|\mathbf{y}\|$, so the operation does not preserve addition.

3. The matrix $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is similar to the matrix $\begin{pmatrix} 2 & 4 \\ 0 & 6 \end{pmatrix}$. `True` `False`

   Solution. False. These two matrices have different determinants, so the matrices cannot be similar matrices.

4. If A is a 3×3 matrix of rank 2, then the dimension of the null space of A^T (the transpose) is equal to 2. `True` `False`

   Solution. False. Since A and A^T have the same rank, the rank of A^T is equal to 2, so by the rank-nullity theorem, the dimension of the null space of A^T is equal to 1.

5. If a 2×2 matrix of real numbers has purely imaginary eigenvalues, then the determinant of the matrix is negative. `True` `False`

   Solution. False. The eigenvalues will be complex conjugates, say $\pm ib$ for some real number b, so the determinant, which is equal to the product of the eigenvalues, will be b^2, which cannot be negative.

In **problems 6–9**, fill in the blanks. (7 points per problem)

Linear Algebra

6. If L is the linear operator on R^2 that doubles the length of each vector and also rotates each vector by 30° counterclockwise, then the standard matrix representation of L is $\begin{pmatrix} \sqrt{3} & \boxed{} \\ \boxed{} & \boxed{} \end{pmatrix}$.

   Solution. We have that $L\begin{pmatrix}1\\0\end{pmatrix} = 2\begin{pmatrix}\cos 30^\circ\\ \sin 30^\circ\end{pmatrix} = \begin{pmatrix}\sqrt{3}\\1\end{pmatrix}$, and similarly $L\begin{pmatrix}0\\1\end{pmatrix} = 2\begin{pmatrix}-\sin 30^\circ\\ \cos 30^\circ\end{pmatrix} = \begin{pmatrix}-1\\ \sqrt{3}\end{pmatrix}$, so the matrix is $\begin{pmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{pmatrix}$.

7. If the scalar product of two vectors in R^3 is equal to 0, then the two vectors are said to be ____________________ .

   Solution. Two vectors whose scalar product equals 0 are *orthogonal*.

8. When $b = \boxed{}$, the linear system $\left\{\begin{aligned} 1x_1 + 2x_2 &= 5 \\ 2x_1 + bx_2 &= 0 \end{aligned}\right\}$ has $x_1 = -1$ and $x_2 = 1$ as a solution in the sense of least squares.

   Solution. Method 1 (sneaky). Since the values $x_1 = -1$ and $x_2 = 1$ evidently do not satisfy the first of the two equations, the least squares solution is not a solution in the ordinary sense. Therefore the indicated system of two equations must be inconsistent. Hence the two columns of the matrix $\begin{pmatrix}1 & 2\\ 2 & b\end{pmatrix}$ do not span R^2. Consequently, the two columns must be proportional, so b must be equal to 4.

   Method 2 (routine). Our method for finding a least squares solution of the system

$$\begin{pmatrix}1 & 2\\ 2 & b\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}5\\0\end{pmatrix}$$

   is to multiply by the transpose matrix, which happens in this example to be equal to the original matrix. The new system is

$$\begin{pmatrix}5 & 2+2b\\ 2+2b & 4+b^2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}5\\10\end{pmatrix}.$$

Substituting the given solution $x_1 = -1$ and $x_2 = 1$ gives the pair of equations

$$\begin{aligned} 2b - 3 &= 5 \\ b^2 - 2b + 2 &= 10. \end{aligned}$$

The first equation implies that $b = 4$ (and this value also satisfies the second equation).

9. Suppose a linear transformation $L : R^2 \to R^2$ has the standard matrix representation $\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$. If $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then the matrix representation of L with respect to the basis $[\mathbf{u}_1, \mathbf{u}_2]$ is $\begin{pmatrix} \boxed{} & 0 \\ \boxed{} & \boxed{} \end{pmatrix}$.

   Solution. Since $L\mathbf{u}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{u}_1$ and $L\mathbf{u}_2 = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5\mathbf{u}_2$, the matrix representation of L with respect to the basis $[\mathbf{u}_1, \mathbf{u}_2]$ is $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$. In other words, the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors of L.

In **problems 10–12**, show your work and explain your method. Continue on the back if you need more space. (15 points each)

10. Suppose $A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 4 & -4 & 5 & -5 \end{pmatrix}$. Find an orthonormal basis for the null space of the matrix A.

    Solution. First find some basis for the null space via Gaussian elimination:

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 4 & -4 & 5 & -5 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 4R_1} \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$
$$\xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right).$$

    In the reduced echelon form, the variables x_2 and x_4 are free variables. One way to get a basis for the null space is first to set $x_4 = 0$ and

$x_2 = 1$, which implies that $x_3 = 0$ and $x_1 = 1$, giving the vector $\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$, and then to set $x_2 = 0$ and $x_4 = 1$, which similarly leads to the vector $\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$. These two vectors form a basis for the null space.

The two vectors happen already to be orthogonal to each other, so all that needs to be done is to divide each by its length to obtain the following orthonormal basis for the null space:

$$\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}.$$

The answer is not unique: in a two-dimensional space, there are infinitely many choices of an orthonormal basis.

11. Suppose $A = \begin{pmatrix}7 & 1 & -4\\4 & 4 & -4\\0 & 0 & 0\end{pmatrix}$. Find a diagonal matrix that is similar to A.

Solution. First find the eigenvalues of A. The characteristic polynomial is

$$\begin{vmatrix}7-\lambda & 1 & -4\\4 & 4-\lambda & -4\\0 & 0 & -\lambda\end{vmatrix} = -\lambda\begin{vmatrix}7-\lambda & 1\\4 & 4-\lambda\end{vmatrix} = -\lambda(\lambda^2 - 11\lambda + 24),$$

which factors as $-\lambda(\lambda-3)(\lambda-8)$. Thus the eigenvalues are 0, 3, and 8.

It is not necessary to compute the corresponding eigenvectors, because the three eigenvectors corresponding to different eigenvalues are linearly independent and therefore form a basis for R^3. If the matrix A represents a linear transformation of R^3 with respect to the standard basis, then the matrix representation of the same transformation with respect to the eigenvector basis will be diagonal with the eigenvalues on the diagonal, and this diagonal matrix will be similar to the matrix A.

The required diagonal matrix is either

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

or one of the other five matrices obtained by permuting the diagonal elements. (The six possible answers correspond to the six different orderings of the eigenvector basis.)

12. Consider the inner product space of continuous functions on the interval $[-1, 1]$, where $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$. Find the projection of the function x^2 onto the subspace spanned by the two functions 1 and x.

    Solution. The integral of an odd function over the symmetric interval $[-1, 1]$ is equal to 0, so $\langle x, x^2\rangle = \int_{-1}^{1} x^3\,dx = 0$. Since the function x^2 is orthogonal to the function x, the projection of the function x^2 onto the subspace spanned by 1 and x is the same as the projection of x^2 onto the function 1: namely,

$$\frac{\langle x^2, 1\rangle}{\|1\|^2}1.$$

    Now $\langle x^2, 1\rangle = \int_{-1}^{1} x^2\,dx = 2\int_{0}^{1} x^2\,dx = 2/3$, and $\|1\|^2 = \langle 1, 1\rangle = \int_{-1}^{1} 1\,dx = 2$. Therefore the required projection equals $(2/3)/2$ or $1/3$.