

Linear Algebra

1. Find values of a , b , and c for which the matrix $\begin{pmatrix} a & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & b & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & c \end{pmatrix}$ is an orthogonal matrix.

Solution. If you look at this problem the right way, you should be able to write down the answer by inspection. The inner product of the first two columns is $a \cdot \frac{1}{\sqrt{3}} + 0 \cdot b - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}}$, so the first two columns are orthogonal only when $a = \frac{1}{\sqrt{2}}$. Knowing the value of a , you can then see by taking the inner product of the first and third columns that $c = \frac{1}{\sqrt{6}}$. Finally, taking the inner product of the second and third columns reveals that $b = -\frac{1}{\sqrt{3}}$.

There are other possible approaches. For instance, since the first column has length equal to 1, you can deduce that $a = \pm \frac{1}{\sqrt{2}}$. Similarly, the second column has length 1, so $b = \pm \frac{1}{\sqrt{3}}$; and the third column has length 1, so $c = \pm \frac{1}{\sqrt{6}}$. You can then determine the correct choice of signs by taking inner products of columns (or of rows). Alternatively, you could multiply the matrix by its transpose, set the result equal to the identity matrix, and solve the resulting equations for a , b , and c .

2. Suppose vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 form an orthonormal basis for a certain inner product space V . Determine the angle between the vectors $\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_1 + \mathbf{v}_3$.

Solution. The angle θ satisfies the following equation:

$$\langle \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3 \rangle = \|\mathbf{v}_1 - \mathbf{v}_2\| \|\mathbf{v}_1 + \mathbf{v}_3\| \cos \theta.$$

By linearity of the inner product, the left-hand side expands to

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle + \langle \mathbf{v}_1, \mathbf{v}_3 \rangle - \langle \mathbf{v}_2, \mathbf{v}_1 \rangle - \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 1 + 0 - 0 - 0 = 1.$$

To compute the right-hand side, observe that

$$\langle \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle - \langle \mathbf{v}_1, \mathbf{v}_2 \rangle - \langle \mathbf{v}_2, \mathbf{v}_1 \rangle + \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 2$$

by orthonormality, so $\|\mathbf{v}_1 - \mathbf{v}_2\| = \sqrt{2}$. An analogous calculation shows that $\|\mathbf{v}_1 + \mathbf{v}_3\| = \sqrt{2}$. Consequently, $1 = 2 \cos \theta$, or $\theta = \cos^{-1} \frac{1}{2}$. Your calculator will give you a numerical answer, but the angle is a standard one that you should know: $\theta = \frac{\pi}{3}$ radians or 60° .