Linear Algebra

1. Find values of a and b for which the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix} 1 & a \\ 2 & b \end{pmatrix}$ with eigenvalue 5.

   Solution. The given information says that

   $$\begin{pmatrix} 1 & a \\ 2 & b \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5 \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

   Equivalently,

   $$\begin{aligned} 3 + 4a &= 15, \\ 6 + 4b &= 20. \end{aligned}$$

   Solving each of these equations shows that $a = 3$ and $b = 7/2$.

2. In the space $C[0,1]$ with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$, use the Gram-Schmidt procedure to find an orthonormal basis for the subspace spanned by the functions 1 and x.

   Solution. Since $\|1\|^2 = \langle 1, 1 \rangle = \int_0^1 1^2\,dx = 1$, the function 1 is already normalized. Now $\langle x, 1 \rangle = \int_0^1 x\,dx = \frac{1}{2}$, so the function $x - \frac{1}{2}$ is orthogonal to the function 1. It remains to normalize the function $x - \frac{1}{2}$.

   Since $\|x - \frac{1}{2}\|^2 = \int_0^1 (x - \frac{1}{2})^2\,dx = \int_0^1 (x^2 - x + \frac{1}{4})\,dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$, the normalized function equals $\sqrt{12}\,(x - \frac{1}{2})$, or $\sqrt{3}\,(2x - 1)$.

   Thus the required orthonormal basis is $[1, \sqrt{3}\,(2x - 1)]$.