Linear Algebra

1. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 5 & 3 & 10 \\ 0 & 18 & 4 \end{pmatrix}$, find a lower-triangular matrix L and an upper-triangular matrix U such that $A = LU$.

Solution. Let E_1 denote the elementary matrix $\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which implements the row operation of subtracting 5 times the first row from the second row. Then

$$E_1 A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 18 & 4 \end{pmatrix}.$$

Similarly, let E_2 denote the elementary matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix}$, which implements the row operation of subtracting 6 times the second row from the third row. Then

$$E_2 E_1 A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Consequently, we may take U to be the upper triangular matrix that appears on the right-hand side: namely,

$$U = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Then $A = E_1^{-1} E_2^{-1} U$, and $L = E_1^{-1} E_2^{-1}$. Explicitly,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix}.$$

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Remarks on the first problem

(a) You can easily check the answer by multiplying L times U to verify that you recover the original matrix A.

(b) The answer is not unique. You can get another valid answer by multiplying L on the right by a diagonal matrix, say $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, and simultaneously multiplying U on the left by the inverse diagonal matrix $\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}$. Thus another answer is:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 10 & 3 & 0 \\ 0 & 18 & 5 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 4/5 \end{pmatrix}.$$

2. Find the value of a for which $\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & a \end{pmatrix} = 0.$

Solution. There are various ways to show that $a = 41$. Here are two methods.

Method 1 Use elementary row operations to bring the matrix to triangular form, and use that the determinant of a triangular matrix equals the product of the diagonal elements:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & a \end{vmatrix} \overset{R_1 \to R_1 - 2R_2}{=} \begin{vmatrix} 1 & 0 & -7 & -8 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & a \end{vmatrix} \overset{R_1 \to R_1 + 7R_3}{=} \begin{vmatrix} 1 & 0 & 0 & 41 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & a \end{vmatrix}$$

$$\overset{R_4 \to R_4 - R_1}{=} \begin{vmatrix} 1 & 0 & 0 & 41 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & a-41 \end{vmatrix} = a - 41.$$

Thus the determinant is equal to 0 when $a = 41$.

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Method 2 Use a cofactor expansion to compute the determinant, expanding along the bottom row:

$$\begin{aligned}\begin{vmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 5 & 6\\ 0 & 0 & 1 & 7\\ 1 & 0 & 0 & a\end{vmatrix} &= -\begin{vmatrix}2 & 3 & 4\\ 1 & 5 & 6\\ 0 & 1 & 7\end{vmatrix} + a\begin{vmatrix}1 & 2 & 3\\ 0 & 1 & 5\\ 0 & 0 & 1\end{vmatrix}\\ &= -\left(0\begin{vmatrix}3 & 4\\ 5 & 6\end{vmatrix} - \begin{vmatrix}2 & 4\\ 1 & 6\end{vmatrix} + 7\begin{vmatrix}2 & 3\\ 1 & 5\end{vmatrix}\right) + a\cdot 1\\ &= -(0-(12-4)+7(10-3))+a = a-41.\end{aligned}$$

As before, we see that the determinant is equal to 0 when $a = 41$.