Linear Algebra

1. The five vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}, \quad \mathbf{x}_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

span R^3. Find three of these vectors that form a basis for R^3. (This is exercise 10 on page 151 of the textbook.)

Solution. In a three-dimensional vector space, any three linearly independent vectors form a basis. One way to solve the problem is by trial and error: pick three of the given vectors, test them for linear independence (for instance, by writing the vectors as the columns of a matrix and checking to see if the determinant of the matrix is nonzero), and then try a different set of three if the first set does not work. If you test first $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$, then $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4]$, and then $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5]$, you will succeed on the third try.

The systematic method is to perform Gaussian elimination on the matrix whose columns are the given vectors:

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 5 & 3 & 7 & 1 \\ 2 & 4 & 2 & 4 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$
$$\xrightarrow{R_3 \to R_3/(-2)} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The interpretation of the calculation is that the first, second, and fifth columns are linearly independent vectors (since the first, second, and fifth variables are the lead variables). Therefore $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5]$ is a basis for R^3.

The answer is not unique. Closer scrutiny shows that $\mathbf{x}_5$ together with *any two* of the other vectors also forms a basis for R^3.

Linear Algebra

2. Find the coordinates of the vector $\begin{pmatrix}1\\1\end{pmatrix}$ with respect to the ordered basis $\left[\begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}\right]$. In other words, find numbers c_1 and c_2 such that

$$\begin{pmatrix}1\\1\end{pmatrix} = c_1\begin{pmatrix}5\\3\end{pmatrix} + c_2\begin{pmatrix}3\\2\end{pmatrix}.$$

(This is part of exercise 4 on page 161 of the textbook.)

Solution. You can solve the problem from first principles, as on the first day of class. You are solving a pair of simultaneous equations for two unknowns, so eliminate one of the variables and proceed.

The thematic method, however, is to say that the matrix $\begin{pmatrix}5 & 3\\3 & 2\end{pmatrix}$ is the transition matrix from the indicated nonstandard basis to the standard basis, so the inverse matrix $\begin{pmatrix}5 & 3\\3 & 2\end{pmatrix}^{-1}$ is the transition matrix from the standard basis to the nonstandard basis. Therefore

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}5 & 3\\3 & 2\end{pmatrix}^{-1}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2 & -3\\-3 & 5\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}.$$

In other words, $c_1 = -1$ and $c_2 = 2$.