Linear Algebra

1. Use Gauss-Jordan reduction to bring the matrix

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 6 & 7 & -2 & 10 \\ 7 & 8 & -4 & 5 \end{pmatrix}$$

to *reduced* echelon form.

Solution. Add -6 times the first row to the second row and -7 times the first row to the third row to obtain the matrix

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 4 & 16 \\ 0 & 1 & 3 & 12 \end{pmatrix}.$$

Subtract the second row from both the third row and the first row to obtain the matrix

$$\begin{pmatrix} 1 & 0 & -5 & -17 \\ 0 & 1 & 4 & 16 \\ 0 & 0 & -1 & -4 \end{pmatrix}.$$

Add 4 times the third row to the second row and -5 times the third row to the first row, and then multiply the third row by -1 to obtain the final answer:

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}.$$

Linear Algebra

2. For which value(s) of the parameter a does the linear system

$$\begin{aligned} 2x_2 - 2x_3 &= 3 \\ -6x_1 + 8x_2 + x_3 &= 0 \\ 2x_1 \qquad - ax_3 &= 4 \end{aligned}$$

have infinitely many solutions for (x_1, x_2, x_3)?

Solution. One way to proceed is to move the bottom equation to the top and divide it by 2 to obtain the equivalent system

$$\begin{aligned} x_1 \qquad - \tfrac{1}{2}ax_3 &= 2 \\ 2x_2 - 2x_3 &= 3 \\ -6x_1 + 8x_2 + x_3 &= 0. \end{aligned}$$

Now add 6 times the first equation to the third equation to obtain the equivalent system

$$\begin{aligned} x_1 \quad - \quad \tfrac{1}{2}ax_3 &= 2 \\ 2x_2 - \quad 2x_3 &= 3 \\ 8x_2 + (1-3a)x_3 &= 0. \end{aligned}$$

Subtracting 4 times the second equation from the third equation gives the equivalent system

$$\begin{aligned} x_1 \quad - \quad \tfrac{1}{2}ax_3 &= 2 \\ 2x_2 - \quad 2x_3 &= 3 \\ (9-3a)x_3 &= 0. \end{aligned}$$

If $a \neq 3$, then the third equation implies that $x_3 = 0$. Back substitution shows that $x_2 = \frac{3}{2}$ and $x_1 = 2$; hence the system has a unique solution.

On the other hand, if $a = 3$, then the third equation disappears, and the system has infinitely many solutions: namely, $x_1 = 2 + \frac{3}{2}x_3$, $x_2 = \frac{3}{2} + x_3$, and the free variable x_3 is arbitrary.

Thus the special value of the parameter a is 3.