Math 304

Linear Algebra

Harold P. Boas

boas@tamu.edu

June 23, 2008

Highlights

From last time:

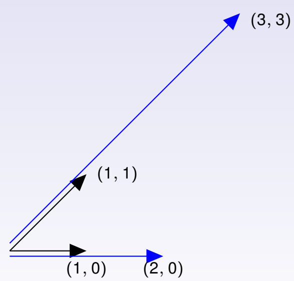
- Gram-Schmidt orthonormalization process and QR factorization

Today:

- eigenvalues and eigenvectors

Example

The matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ defines a linear transformation of R^2 that is easy to understand. The transformation stretches the vector $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by a factor of 2 and stretches the vector $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by a factor of 3.



The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are called *eigenvectors*, and the scale factors 2 and 3 are the corresponding *eigenvalues*.

The transformation is particularly simple to describe in the basis $[\mathbf{u}_1, \mathbf{u}_2]$: namely, the matrix $U^{-1}AU$ is the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Computing eigenvectors

Example. The matrix $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$ has 3 as one of its eigenvalues. Find a corresponding eigenvector.

Solution. We seek a vector $\mathbf{v}$ such that $A\mathbf{v} = 3\mathbf{v}$. Equivalently, $\mathbf{v}$ should be in the nullspace of the matrix $A - 3I$ (I = identity). Find the nullspace by row reduction:

$$\left(\begin{array}{ccc|c} 9 & 4 & -5 & 0 \\ -8 & -3 & 5 & 0 \\ 10 & 4 & -6 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -8 & -3 & 5 & 0 \\ 10 & 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 10R_1]{R_2 \to R_2 + 8R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right) \xrightarrow[\text{steps}]{\text{three}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $\mathbf{v} = (1, -1, 1)^T$ is an eigenvector with eigenvalue 3.

Computing eigenvalues

Example. The matrix $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$ has other eigenvalues besides the number 3. Find them.

Solution. The condition for a number λ to be an eigenvalue of A is that the matrix $A - \lambda I$ have a non-trivial nullspace. Equivalently, $\det(A - \lambda I) = 0$, the *characteristic equation*:

$$0 = \begin{vmatrix} 12-\lambda & 4 & -5 \\ -8 & 0-\lambda & 5 \\ 10 & 4 & -3-\lambda \end{vmatrix} \underset{R_1+R_2}{\overset{R_1\to}{=}} \begin{vmatrix} 4-\lambda & 4-\lambda & 0 \\ -8 & -\lambda & 5 \\ 10 & 4 & -3-\lambda \end{vmatrix}$$

$$\underset{C_2-C_1}{\overset{C_2\to}{=}} \begin{vmatrix} 4-\lambda & 0 & 0 \\ -8 & 8-\lambda & 5 \\ 10 & -6 & -3-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 8-\lambda & 5 \\ -6 & -3-\lambda \end{vmatrix}$$

$$= (4-\lambda)(\lambda^2 - 5\lambda + 6) = (4-\lambda)(\lambda-3)(\lambda-2).$$

Therefore the eigenvalues of A are 4, 3, and 2.

Eigenvalues and similarity

If A and B are similar matrices ($B = S^{-1}AS$), then A and B have the same eigen*values* (but not the same eigen*vectors*). Here's why. If $A\mathbf{v} = \lambda\mathbf{v}$, then $B\mathbf{w} = \lambda\mathbf{w}$ with $\mathbf{w} = S^{-1}\mathbf{v}$. In fact, $B\mathbf{w} = (S^{-1}AS)(S^{-1}\mathbf{v}) = S^{-1}A\mathbf{v} = S^{-1}\lambda\mathbf{v} = \lambda\mathbf{w}$.

Example. Since the matrix $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$ has eigenvalues 4, 3, and 2, the matrix A is similar to a diagonal matrix $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Similar matrices have equal determinants, so $\det(A) = 24$ (the product of the eigenvalues).
Similar matrices have equal traces too, and indeed $12 + 0 - 3 = 4 + 3 + 2$.