Math 304

Linear Algebra

Harold P. Boas

boas@tamu.edu

June 24, 2008

Highlights

From last time:

- eigenvalues and eigenvectors

Today:

- application of eigenvectors to systems of differential equations

Reminders on first-order differential equations

Example. Solve the differential equation $\frac{dy}{dt} = 3y$ or $y' = 3y$ with the initial condition $y(0) = 4$.

Solution. We know a function whose derivative is 3 times the function: namely, $y(t) = e^{3t}$, or more generally $y(t) = ce^{3t}$ for an arbitrary constant c.

Thus the *general solution* to $y' = 3y$ is $y(t) = ce^{3t}$.

The *particular solution* satisfying the initial condition $y(0) = 4$ is $y(t) = 4e^{3t}$.

Linear systems of differential equations

Exercise 1(b), page 323

Find the general solution to the system $\begin{cases} y_1' = 2y_1 + 4y_2 \\ y_2' = -y_1 - 3y_2. \end{cases}$

Solution. Write $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}$. Then the system says $\mathbf{y}' = A\mathbf{y}$.

If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$ is a solution of the differential equation. Since A has eigenvalues 1 and -2 with corresponding eigenvectors $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, the general solution is the superposition

$$\mathbf{y}(t) = c_1 e^t \begin{pmatrix} 4 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ or } \begin{cases} y_1(t) = 4c_1 e^t + c_2 e^{-2t} \\ y_2(t) = -c_1 e^t - c_2 e^{-2t}. \end{cases}$$

An initial condition for $\mathbf{y}(0)$ would let you determine c_1 and c_2.

Euler's formula

The complex exponential function is related to the trigonometric functions via *Euler's formula*:

$$e^{it} = \cos(t) + i\sin(t).$$

For example, $e^{i\pi} = -1$, $e^{i\pi/4} = (1+i)/\sqrt{2}$, and $e^{(2+3i)t} = e^{2t}(\cos(3t) + i\sin(3t))$.

Higher-order systems

Example: exercise 5(b), page 324

Solve $\begin{cases} y_1'' = 2y_1 + y_2' \\ y_2'' = 2y_2 + y_1'. \end{cases}$

Solution strategy. We know how to handle systems of *first-order* differential equations, so introduce two new variables y_3 and y_4 via $y_1' = y_3$ and $y_2' = y_4$. The system becomes

$$\begin{aligned} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= 2y_1 + y_4 \\ y_4' &= 2y_2 + y_3 \end{aligned} \qquad \text{or} \qquad \mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \mathbf{y}.$$

Proceed as before: find the eigenvalues [± 1 and ± 2], the corresponding eigenvectors, and write the general solution [with four arbitrary constants c_1, c_2, c_3, and c_4].

Differential equations with complex eigenvalues

Example: exercise 1(d), page 323

Solve $\begin{cases} y_1' = y_1 - y_2 \\ y_2' = y_1 + y_2 \end{cases}$ or $\mathbf{y}' = A\mathbf{y}$ with $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

The characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$. By the quadratic formula, $\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. An eigenvector corresponding to eigenvalue $1+i$ is $\begin{pmatrix} i \\ 1 \end{pmatrix}$. One complex-valued solution is

$$\mathbf{y}(t) = e^{(1+i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} e^t(-\sin(t) + i\cos(t)) \\ e^t(\cos(t) + i\sin(t)) \end{pmatrix}.$$

The differential equation is real-valued, so the real part and the imaginary part of the complex solution both are real solutions. The general (real) solution is therefore

$$\mathbf{y}(t) = c_1 e^t \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}.$$