Linear Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Find the orthogonal complement of the subspace of R^3 spanned by the two vectors $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\2\end{pmatrix}$.

   [This is exercise 3(b) on page 233 of the textbook.]

   Solution. The two given vectors span a two-dimensional subspace of R^3, so the orthogonal complement is a one-dimensional subspace of R^3. To identify this one-dimensional subspace, we need to find a (nontrivial) vector that is orthogonal to both of the given vectors.

   Method 1 We can solve the problem by finding the null space of the 2×3 matrix that has the given vectors as rows. Use Gaussian elimination:

$$\left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 1 & -1 & 2 & 0\end{array}\right) \xrightarrow{R2\to R2-R1} \left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 0 & -3 & 1 & 0\end{array}\right)$$
$$\xrightarrow{R2\to(-1/3)R2} \left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\ 0 & 1 & -1/3 & 0\end{array}\right) \xrightarrow{R1\to R1-2R2} \left(\begin{array}{ccc|c}1 & 0 & 5/3 & 0\\ 0 & 1 & -1/3 & 0\end{array}\right).$$

   The deduction from this calculation is that x_3 is a free variable, $x_2 = (1/3)x_3$, and $x_1 = -(5/3)x_3$. Therefore vectors in the null space have the form $\begin{pmatrix}-(5/3)x_3\\(1/3)x_3\\x_3\end{pmatrix}$, where x_3 is arbitrary. In other words, the vector $\begin{pmatrix}-5/3\\1/3\\1\end{pmatrix}$ is a basis for the orthogonal complement that we seek. Any nontrivial scalar multiple of this vector would do as well, so we could clear the fractions and use either the vector $\begin{pmatrix}-5\\1\\3\end{pmatrix}$ or its negative as a basis vector of our one-dimensional subspace.

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Method 2 Since the problem is set in R^3, there is a special device available to find a vector orthogonal to two given vectors. You learned in calculus class about the vector cross product, which produces a particular vector orthogonal to two given vectors. Using the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, you get the cross product by computing the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

via cofactor expansion across the top row. The result is $5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. The corresponding vector is $\begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$, a scalar multiple of the vector found by Method 1.

Remark The cross product exists only in R^3, so if you had an analogous problem in R^4, then you would have to use Method 1.

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2. Find a least-squares solution to the inconsistent linear system

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 0 \\ x_1 + 0x_2 &= 4. \end{aligned}$$

Solution. The corresponding matrix equation is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}.$$

Multiplying both sides by the transpose matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ gives the consistent linear system

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}.$$

Since the matrix on the left-hand side is diagonal, you can immediately read off the unique least-squares solution $x_1 = 7/3$ and $x_2 = 3/2$.