Linear Algebra

Instructions Please write your name in the upper right-hand corner of the page. Circle the correct answer in each of the following items.

1. The set of vectors $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ forms an orthonormal basis for R^2.
   `True` `False`

   Solution. False. The vectors are orthogonal to each other, but they are not normalized to length 1.

2. In the space $C[-\pi, \pi]$ of continuous functions on the interval $[-\pi, \pi]$ equipped with the inner product
$$\langle f(x), g(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx,$$
   the functions $\sin(x)$ and $\cos(x)$ are orthogonal. `True` `False`

   Solution. True. This orthogonality is a basic example discussed in class and in the textbook on page 265. To verify that
$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x)\cos(x)\, dx = 0,$$
   observe that $\frac{1}{2}(\sin(x))^2$ is an anti-derivative of $\sin(x)\cos(x)$.

3. In a vector space that is equipped with a norm, the distance between vectors $\mathbf{x}$ and $\mathbf{y}$ is defined to be $\|\mathbf{y} - \mathbf{x}\|$. `True` `False`

   Solution. True. The statement is the boxed definition on page 252 of the textbook.

4. Every orthogonal matrix is invertible. `True` `False`

   Solution. True. A property equivalent to the definition of orthogonal matrix is that the transpose matrix is equal to the inverse matrix; hence the inverse matrix must exist! See the box on page 259 of the textbook.

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5. For every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space, the following inequality holds: $|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\|$. `True` `False`

Solution. True. This inequality is the famous Cauchy-Schwarz inequality. The inequality is what makes it sensible to define the angle θ between two vectors by the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos(\theta).$$

See Theorem 5.4.2 on page 249 of the textbook.