Linear Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Fill in the four blanks in the following matrix product:

$$\begin{pmatrix} \square & 0 & \square \\ \square & 0 & \square \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 16 & 23 \\ 0 & 1 \end{pmatrix}.$$

Solution. This problem is simple enough to write down the answer by inspection, but here is a systematic procedure for working out the answer.

Write the first matrix as $\begin{pmatrix} a & 0 & b \\ c & 0 & d \end{pmatrix}$, where the values of a, b, c, and d are to be found. Carrying out the matrix multiplication shows that

$$\begin{pmatrix} 4b & a+5b \\ 4d & c+5d \end{pmatrix} = \begin{pmatrix} 16 & 23 \\ 0 & 1 \end{pmatrix}.$$

Matching the corresponding entries on the two sides gives a linear system of four equations in four unknowns, but the equations are so simple that there is no point in setting up a 4×4 matrix. Comparing the first columns above shows that $b = 4$ and $d = 0$. Comparing the second columns and using the already determined values for b and d shows that $a = 3$ and $c = 1$.

Thus the unknown matrix is $\begin{pmatrix} 3 & 0 & 4 \\ 1 & 0 & 0 \end{pmatrix}$.

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2. Write the vector $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ as a linear combination of the two vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. In other words, find numbers x_1 and x_2 such that

$$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}.$$

Solution. We have to solve a linear system for which the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 3 & -4 \\ 2 & 4 & 6 \end{array}\right).$$

Subtracting twice the first row from the second row gives the matrix

$$\left(\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & -2 & 14 \end{array}\right).$$

From the bottom row, we see that $x_2 = -7$. Substituting back into the first row, we see that $x_1 - 21 = -4$, so $x_1 = 17$. Thus

$$17 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 7 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}.$$