Linear Algebra

Instructions   Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Find all values of t for which

$$\det \begin{pmatrix} 0 & 0 & 0 & t \\ 1 & 0 & t & * \\ 0 & 2 & 0 & * \\ 1 & * & 3 & * \end{pmatrix} = 0.$$

(The asterisks represent unspecified numbers that you do not need to know to solve the problem.)

Solution. We know two methods for computing determinants, and either method will solve this problem.

Method 1 Using the cofactor expansion along the top row, write

$$\det \begin{pmatrix} 0 & 0 & 0 & t \\ 1 & 0 & t & * \\ 0 & 2 & 0 & * \\ 1 & * & 3 & * \end{pmatrix} = -t \det \begin{pmatrix} 1 & 0 & t \\ 0 & 2 & 0 \\ 1 & * & 3 \end{pmatrix}.$$

Now using the cofactor expansion along the middle row, write

$$-t \det \begin{pmatrix} 1 & 0 & t \\ 0 & 2 & 0 \\ 1 & * & 3 \end{pmatrix} = -2t \det \begin{pmatrix} 1 & t \\ 1 & 3 \end{pmatrix} = -2t(3 - t).$$

Therefore the original determinant is equal to 0 when $t = 0$ and when $t = 3$.

Method 2 Use row operations to bring the matrix to triangular form. Reordering the rows at worst changes the sign of the determinant, so

$$\det \begin{pmatrix} 0 & 0 & 0 & t \\ 1 & 0 & t & * \\ 0 & 2 & 0 & * \\ 1 & * & 3 & * \end{pmatrix} = \pm \det \begin{pmatrix} 1 & 0 & t & * \\ 0 & 2 & 0 & * \\ 1 & * & 3 & * \\ 0 & 0 & 0 & t \end{pmatrix}.$$

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(Since we are going to set the determinant equal to 0, we do not need to keep track of the sign.) Subtracting the first row from the third row does not change the determinant, which now equals

$$\pm \det \begin{pmatrix} 1 & 0 & t & * \\ 0 & 2 & 0 & * \\ 0 & * & 3-t & * \\ 0 & 0 & 0 & t \end{pmatrix}.$$

We do not know the value of the matrix entry a_{32}, but we can nonetheless zero out that entry by adding a suitable multiple of the second row to the third row (an operation that does not change the determinant). The determinant now equals

$$\pm \det \begin{pmatrix} 1 & 0 & t & * \\ 0 & 2 & 0 & * \\ 0 & 0 & 3-t & * \\ 0 & 0 & 0 & t \end{pmatrix}.$$

The determinant of a matrix in triangular form is the product of the elements on the diagonal, so we get

$$\pm 2t(3-t).$$

We see as before that the determinant equals 0 when $t = 0$ and when $t = 3$.

2. Fill in the three indicated matrix entries in the following equation that expresses a certain matrix as the product of a lower triangular matrix times an upper triangular matrix:

$$\begin{pmatrix} 1 & 7 & * \\ 3 & 23 & * \\ 0 & 8 & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Box & 1 & 0 \\ \Box & \Box & 1 \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

(The asterisks represent unspecified numbers that you do not need to know to solve the problem.)

Solution. You can solve this problem in two ways, one thematic and one using first principles.

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Method 1   To bring the matrix A on the left-hand side to echelon form, the first step is to subtract 3 times the first row from the second row, giving

$$\begin{pmatrix} 1 & 7 & * \\ 0 & 2 & * \\ 0 & 8 & * \end{pmatrix}.$$

The second step is to subtract 4 times the second row from the third row, giving

$$\begin{pmatrix} 1 & 7 & * \\ 0 & 2 & * \\ 0 & 0 & * \end{pmatrix}.$$

If you understand how the LU factorization algorithm works, you can immediately write down the required lower triangular factor L: it is

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}.$$

Here are more details. Using the elementary matrices that implement the elementary row operations, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 & * \\ 3 & 23 & * \\ 0 & 8 & * \end{pmatrix} = \begin{pmatrix} 1 & 7 & * \\ 0 & 2 & * \\ 0 & 0 & * \end{pmatrix},$$

or $E_2 E_1 A = U$ in symbolic form. Therefore $A = E_1^{-1} E_2^{-1} U$, so

$$L = E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}.$$

Method 2   It is possible to solve the problem using only the knowledge of how matrix multiplication works. Indeed, since the first row of the lower triangular matrix is $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, the first row of the upper triangular matrix must be identical to the first row of the matrix on

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the left-hand side. Therefore our problem takes the form

$$
\begin{pmatrix} 1 & 7 & * \\ 3 & 23 & * \\ 0 & 8 & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \square & 1 & 0 \\ \square & \square & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.
$$

Since $a_{21} = 3$, we can read off the first of the unknown entries to get

$$
\begin{pmatrix} 1 & 7 & * \\ 3 & 23 & * \\ 0 & 8 & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ \square & \square & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.
$$

Since $a_{22} = 23$, we can read off one of the mystery entries in the upper triangular matrix:

$$
\begin{pmatrix} 1 & 7 & * \\ 3 & 23 & * \\ 0 & 8 & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ \square & \square & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 & * \\ 0 & 2 & * \\ 0 & 0 & * \end{pmatrix}.
$$

Since $a_{31} = 0$, we get another of the required entries in the lower triangular matrix:

$$
\begin{pmatrix} 1 & 7 & * \\ 3 & 23 & * \\ 0 & 8 & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \square & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 & * \\ 0 & 2 & * \\ 0 & 0 & * \end{pmatrix}.
$$

Since $a_{32} = 8$, we can read off the final required value:

$$
\begin{pmatrix} 1 & 7 & * \\ 3 & 23 & * \\ 0 & 8 & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 & * \\ 0 & 2 & * \\ 0 & 0 & * \end{pmatrix}.
$$