Linear Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Find a spanning set for the null space of the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 8 \end{pmatrix}$.

   Solution. You can find the null space by bringing the augmented matrix
$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 2 & 1 & 0 & 8 & 0 \end{array}\right)$$
   to row echelon form. Subtracting twice the first row from the second row gives
$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -3 & -6 & 0 & 0 \end{array}\right).$$
   Dividing the second row by -3 gives
$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right).$$
   Subtracting twice the second row from the first row gives
$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 4 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array}\right).$$
   From this reduced row echelon form you can read off that x_3 and x_4 are free variables, $x_2 = -2x_3$, and $x_1 = x_3 - 4x_4$. The null space consists of all vectors of the form
$$\begin{pmatrix} x_3 - 4x_4 \\ -2x_3 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{or} \qquad x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$
   where the free variables x_3 and x_4 can take any value. Therefore a spanning set for the null space is the pair of vectors
$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

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2. In the vector space $C[0,1]$ of continuous functions on the interval $[0,1]$, consider the set of functions f such that $\int_0^1 f(x)\,dx = 1$. Is this sub*set* of $C[0,1]$ a sub*space*? Explain why or why not.

Solution. A subset of a vector space is a subspace if it is closed under addition and under multiplication by scalars (page 124 in the textbook). The indicated set is closed under neither operation, so it is not a subspace.

Indeed, if $\int_0^1 f_1(x)\,dx = 1$ and $\int_0^1 f_2(x)\,dx = 1$, then

$$\int_0^1 [f_1(x) + f_2(x)]\,dx = \int_0^1 f_1(x)\,dx + \int_0^1 f_2(x)\,dx = 1 + 1 = 2 \neq 1,$$

so the function $f_1 + f_2$ does not lie in the set: closure under addition fails. Moreover, if c is any scalar different from 1, and $\int_0^1 f(x)\,dx = 1$, then

$$\int_0^1 cf(x)\,dx = c\int_0^1 f(x)\,dx = c \cdot 1 = c \neq 1,$$

so the function cf does not lie in the set: closure under scalar multiplication fails.