Linear Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Let L be the linear operator on R^3 defined by the property that

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \end{pmatrix}.$$

Find a basis for the kernel of L.

Solution. The kernel consists of those vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ for which

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In other words, $x_1 - x_2 = 0$ and $x_2 - x_3 = 0$ and $x_3 - x_1 = 0$. Consequently, $x_1 = x_2 = x_3$. Thus the vectors in the kernel of L have all three entries equal to each other: such vectors are all scalar multiples of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. In other words, the kernel of L is a one-dimensional subspace of R^3, and the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a basis for the kernel.

Remark You could alternatively determine the kernel by row reducing the augmented matrix:

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

which leads to the same conclusion as before.

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2. Suppose $L\colon P_2 \to P_3$ is the linear transformation defined by the property that $L(p(x)) = xp(x)$ for every polynomial $p(x)$. (Recall that P_n denotes the space of polynomials of degree less than n.) Determine the matrix representation of L with respect to the ordered basis $[1, x]$ in P_2 and the ordered basis $[1, 1+x, 1+x+x^2]$ in P_3.

Solution. To determine the matrix, we examine the images in P_3 of the two basis functions in P_2. Since $L(1) = x$, and x is expressed in terms of the basis in P_3 as the second basis function minus the first, the first column of the matrix is $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Since $L(x) = x^2$, and x^2 is expressed in terms of the basis in P_3 as the third basis function minus the second, the second column of the matrix is $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. Thus the matrix that represents the indicated linear transformation with respect to the given bases is

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}.$$