Math 304

Linear Algebra

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About the final exam

- Final exam takes place in this room 12:30–2:30PM on Friday, May 7.
- Exam is comprehensive (covers the whole course).
- Exam has same style as the two midterm exams.
- The three 15-point work-out problems are selected from the following topics:
  - LU factorization (Section 1.4)
  - Change of basis (Section 3.5)
  - Similar matrices (Section 4.3)
  - Least squares (Section 5.3)
  - QR factorization (Section 5.6)
  - Diagonalization of a matrix (Section 6.3)

Snapshot

Last time:

- Gram-Schmidt orthonormalization process and the *QR* factorization

Today:

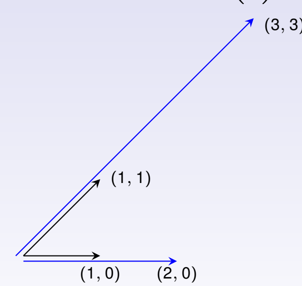
- Eigenvalues and eigenvectors

Next time:

- Applications of eigenvectors to systems of differential equations

Example

The matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ defines a linear transformation of R^2 that is relatively easy to understand. The transformation stretches the vector $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by a factor of 2 and stretches the vector $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by a factor of 3.



The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are called *eigenvectors*, and the scale factors 2 and 3 are the corresponding *eigenvalues*.

The transformation is particularly simple to describe in the basis $[\mathbf{u}_1, \mathbf{u}_2]$ using transition matrix $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: namely, $U^{-1}AU$ is the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Computing eigenvectors

Example. The matrix $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$ has 3 as one of its eigenvalues. Find a corresponding eigenvector.

Solution. We seek $\mathbf{v}$ such that $A\mathbf{v} = 3\mathbf{v}$, or $(A - 3I)\mathbf{v} = 0$, where $I =$ identity matrix. That is, vector $\mathbf{v}$ is in the nullspace of the matrix $A - 3I$. Find the nullspace by row reduction:

$$\left(\begin{array}{ccc|c} 9 & 4 & -5 & 0 \\ -8 & -3 & 5 & 0 \\ 10 & 4 & -6 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -8 & -3 & 5 & 0 \\ 10 & 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 10R_1]{R_2 \to R_2 + 8R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right) \xrightarrow[\text{steps}]{\text{three}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So $\mathbf{v} = (1, -1, 1)^T$ is an eigenvector with eigenvalue 3.

Computing eigenvalues

Example. The matrix $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$ has other eigenvalues besides the number 3. Find them.

Solution. The condition for a number λ to be an eigenvalue of A is that the matrix $A - \lambda I$ has a non-trivial nullspace. Equivalently, $\det(A - \lambda I) = 0$, the *characteristic equation*:

$$0 = \begin{vmatrix} 12-\lambda & 4 & -5 \\ -8 & 0-\lambda & 5 \\ 10 & 4 & -3-\lambda \end{vmatrix} \overset{R_1 \to}{\underset{R_1+R_2}{=}} \begin{vmatrix} 4-\lambda & 4-\lambda & 0 \\ -8 & -\lambda & 5 \\ 10 & 4 & -3-\lambda \end{vmatrix}$$

$$\overset{C_2 \to}{\underset{C_2 - C_1}{=}} \begin{vmatrix} 4-\lambda & 0 & 0 \\ -8 & 8-\lambda & 5 \\ 10 & -6 & -3-\lambda \end{vmatrix} = (4-\lambda) \begin{vmatrix} 8-\lambda & 5 \\ -6 & -3-\lambda \end{vmatrix}$$

$$= (4-\lambda)(\lambda^2 - 5\lambda + 6) = (4-\lambda)(\lambda - 3)(\lambda - 2).$$

Therefore the eigenvalues of A are 4, 3, and 2.

Eigenvalues and similarity

If A and B are similar matrices ($B = S^{-1}AS$), then A and B have the same eigen*values* (but not the same eigen*vectors*). Here's why. If $A\mathbf{v} = \lambda\mathbf{v}$, then $B\mathbf{w} = \lambda\mathbf{w}$ with $\mathbf{w} = S^{-1}\mathbf{v}$. In fact, $B\mathbf{w} = (S^{-1}AS)(S^{-1}\mathbf{v}) = S^{-1}A\mathbf{v} = S^{-1}\lambda\mathbf{v} = \lambda\mathbf{w}$.

Example. Since the matrix $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$ has eigenvalues 4, 3, and 2, the matrix A is similar to a diagonal matrix $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Similar matrices have equal determinants, so $\det(A) = 24$ (the product of the eigenvalues).
Similar matrices have equal traces too, and indeed $12 + 0 - 3 = 4 + 3 + 2$.