Math 304
Linear Algebra

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Snapshot

From last time:

- Application of eigenvalues and eigenvectors to systems of linear differential equations.

Today:

- Diagonalization of matrices and applications.

Next time:

- We will review for the final exam during our last class meeting, which is Thursday, April 29.

  The final exam will be held 12:30–2:30PM on Friday, May 7.

Eigenvector basis $\iff$ diagonal matrix

Suppose a linear operator L on R^3 is represented in a basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ by the diagonal matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

This means that $L\mathbf{u}_1 = 2\mathbf{u}_1$ and $L\mathbf{u}_2 = 3\mathbf{u}_2$ and $L\mathbf{u}_3 = 5\mathbf{u}_3$. In other words, the basis vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are eigenvectors of the operator L.

A square matrix A is *diagonalizable* if the linear transformation $L(\mathbf{x}) = A\mathbf{x}$ can be represented in some basis by a diagonal matrix; in other words, if there is a basis consisting of eigenvectors of A; equivalently, if there is a transition matrix S such that $S^{-1}AS$ is a diagonal matrix; that is, if A is similar to a diagonal matrix.

Example

Diagonalize the matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$. In other words, find an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$ or, equivalently, $A = SDS^{-1}$.

Solution. First find the eigenvalues and eigenvectors of A.

From last time, $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 1, and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue -2. The matrix $S = \begin{bmatrix} 4 & 1 \\ -1 & -1 \end{bmatrix}$ is the transition matrix from the eigenvector basis to the standard basis, and the matrix $S^{-1}AS$ is the diagonal matrix $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

Continuation

If $A = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$, find the power A^{1000}.

Solution. Since $S^{-1}AS = D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, and

$D^{1000} = \begin{bmatrix} 1 & 0 \\ 0 & 2^{1000} \end{bmatrix}$, it follows that $A^{1000} = SD^{1000}S^{-1} =$

$$\begin{bmatrix} 4 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{1000} \end{bmatrix} \left(-\frac{1}{3}\right) \begin{bmatrix} -1 & -1 \\ 1 & 4 \end{bmatrix} =$$

$$\left(-\frac{1}{3}\right) \begin{bmatrix} -4 + 2^{1000} & -4 + 4 \times 2^{1000} \\ 1 - 2^{1000} & 1 - 4 \times 2^{1000} \end{bmatrix}.$$

More. Since the exponential function is given by a power series ($e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$), define e^A via

$$e^A := I + A + \frac{1}{2!}A^2 + \cdots = Se^DS^{-1} = S \begin{bmatrix} e^1 & 0 \\ 0 & e^{-2} \end{bmatrix} S^{-1}$$

$$= \left(-\frac{1}{3}\right) \begin{bmatrix} -4e + e^{-2} & -4e + 4e^{-2} \\ e - e^{-2} & e - 4e^{-2} \end{bmatrix}.$$

Application to differential equations

You have two ways to solve the system of differential equations $\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \mathbf{y}$.

(a) From last time, you can write the general solution as $\mathbf{y}(t) = c_1 e^t \begin{bmatrix} 4 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) With a different choice of c_1 and c_2, you can write

$$\mathbf{y}(t) = e^{tA} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Se^{tD}S^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} =$$

$$\left(-\frac{1}{3}\right) \begin{bmatrix} -4e^t + e^{-2t} & -4e^t + 4e^{-2t} \\ e^t - e^{-2t} & e^t - 4e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

In the second form, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$.