Linear Algebra

Write your **name**: Answer Key (2 points).

In **problems 1–5**, circle the correct answer. (5 points per problem)

1. When A and B are matrices of size 4×6, the sum matrix $A + B$ is always equal to $B + A$. `True` `False`

   Solution. True: Matrix addition is commutative. (The noncommutative operation is matrix *multiplication*.) The size of the matrices is irrelevant, as long as A and B have the same shape (ensuring that the sum makes sense).

2. If A is an invertible $n \times n$ matrix, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.
   `True` `False`

   Solution. True: The determinant is a multiplicative function, so
   $$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$
   Therefore $\det(A)$ and $\det(A^{-1})$ are reciprocals.

3. For every $n \times n$ matrix A, the product matrix $A^T A$ is a symmetric matrix. `True` `False`

   Solution. True: A matrix is symmetric if and only if it equals its transpose. Now the transpose of a product is the product of the transposes in reverse order, so
   $$\left(A^T A\right)^T = A^T (A^T)^T = A^T A.$$
   Thus the matrix $A^T A$ is indeed equal to its transpose.

4. If A is a 4×3 matrix, and one of the columns of A has all its entries equal to 0, then the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. `True` `False`

   Solution. True: If the first column (for instance) has all its entries equal to 0, then every scalar multiple of the vector $(1, 0, 0)^T$ is in the null space of the matrix. Hence the homogeneous linear system has infinitely many solutions. Similar reasoning applies if one of the other columns has all its entries equal to 0.

5. The polynomials $1+x$, $1-x$, and $x+x^2$ form a spanning set for the vector space P_3 (which consists of polynomials of degree less than or equal to 2). `True` `False`

Solution. True: It suffices to show that the polynomials 1, x, and x^2 are in the span of the given polynomials. Now $1 = \frac{1}{2}[(1+x)+(1-x)]$, and $x = \frac{1}{2}[(1+x)-(1-x)]$, and $x^2 = \frac{1}{2}[(1-x)-(1+x)]+(x+x^2)$.

Remark. If the true/false values were chosen at random, the probability of all five being true would be 1/32, or about 3%: not very likely, but not out of the question.

In **problems 6–9**, fill in the blanks. (7 points per problem)

6. If $A = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ \square & 0 & 1 \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} 1 & 0 & \square \\ 0 & 1 & 0 \\ \square & 0 & 1 \end{pmatrix}$.

Solution. The answer is

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A^{-1} = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

How to find it? Since AA^{-1} equals the identity matrix, using the rule for matrix multiplication to find the $(1,1)$ entry of the product shows that the missing entry in the lower left-hand corner of A^{-1} is 0. Thus A^{-1} is upper triangular. Therefore A is upper triangular too, so its missing entry is 0. Consequently, the matrix A is an elementary matrix corresponding to the row operation of adding 5 times the third row to the first row. Therefore A^{-1} is the inverse elementary matrix corresponding to subtracting 5 times the third row from the first row.

7. If $A = \begin{pmatrix} \square & 5 \\ \square & \square \end{pmatrix}$, then $A + A^T = \begin{pmatrix} 0 & 7 \\ \square & 0 \end{pmatrix}$.

Solution. The answer is $A = \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix}$ and $A + A^T = \begin{pmatrix} 0 & 7 \\ 7 & 0 \end{pmatrix}$.

How to find it? The matrix $A + A^T$ evidently is equal to its transpose, so the missing entry in this sum matrix is 7. Now A and A^T have the same diagonal entries, so the diagonal entries of $A + A^T$ are twice the corresponding entries of A. Since $A + A^T$ has 0 on the diagonal, so must A. The remaining missing entry in A has to be 2 to make $A + A^T$ have the indicated form.

8. A linearly independent spanning set for a vector space is called a

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Solution. A linearly independent spanning set is a *basis*.

9. The linear system $\begin{cases} x_1 - 2x_2 + 3x_3 - 4x_4 = 5 \\ 2x_1 - 4x_2 + 6x_3 - 8x_4 = 10 \end{cases}$ has how many solutions? ________________________________
(none, one, or infinitely many?)

Solution. The second equation is twice the first equation, so the system effectively reduces to a single equation with three free variables. Hence there are *infinitely many* solutions.

Linear Algebra

In **problems 10–12**, show your work and explain your method.
(15 points per problem)

10. Determine the null space of the matrix $\begin{pmatrix} 0 & 2 & 1 & 8 \\ 2 & 0 & 1 & 0 \end{pmatrix}$.

    Solution. Set up an augmented matrix and use Gaussian elimination:

$$\left(\begin{array}{cccc|c} 0 & 2 & 1 & 8 & 0 \\ 2 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R1 \leftrightarrow R2} \left(\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 8 & 0 \end{array}\right)$$
$$\xrightarrow[R2 \to \frac{1}{2}R2]{R1 \to \frac{1}{2}R1} \left(\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 4 & 0 \end{array}\right).$$

    From this reduced row-echelon form, you can read off that $x_1 = -\frac{1}{2}x_3$, and $x_2 = -\frac{1}{2}x_3 - 4x_4$. Therefore the vectors $(x_1, x_2, x_3, x_4)^T$ in the null space have the form

$$\begin{pmatrix} -\frac{1}{2}x_3 \\ -\frac{1}{2}x_3 - 4x_4 \\ x_3 \\ x_4 \end{pmatrix}, \qquad \text{or} \qquad x_3 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -4 \\ 0 \\ 1 \end{pmatrix},$$

    where x_3 and x_4 can take arbitrary values. In other words, the null space is the span of the vectors $(1, 1, -2, 0)^T$ and $(0, -4, 0, 1)^T$.

11. Determine a value of x for which

$$\det \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & 0 & 0 & 8 \end{pmatrix} = 18.$$

    Solution.

$$\det \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & 0 & 0 & 8 \end{pmatrix} \overset{R1 \leftrightarrow R4}{=} -\det \begin{pmatrix} -x & 0 & 0 & 8 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \overset{\Delta \text{ form}}{=} 2x^2.$$

    Thus $2x^2 = 18$, or $x^2 = 9$, so $x = \pm 3$.

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12. The equation below shows an LU factorization. Fill in the missing entries, and explain your strategy for finding them.

$$\begin{pmatrix} \square & 5 & 0 \\ 3 & \square & 2 \\ 0 & 4 & \square \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & \square & \square \\ 0 & 1 & \square \\ 0 & 0 & 1 \end{pmatrix}$$

Solution. Here is the answer:

$$\begin{pmatrix} 1 & 5 & 0 \\ 3 & 16 & 2 \\ 0 & 4 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

How to find it? The rule for matrix multiplication shows that the entry in the upper left-hand corner of the matrix on the left-hand side equals 1. If you understand how the LU factorization algorithm works, you can now immediately fill in the missing entries in the matrix on the right-hand side. The rule for matrix multiplication then forces the remaining missing entries on the diagonal of the matrix on the left-hand side.

Actually, you can solve the problem without knowing anything about the LU algorithm. If you put letters in the missing boxes, multiply out the matrix product, and interpret the matrix equation as a linear system in the unknown letters, then you can solve by Gaussian elimination. This method takes a little longer but reaches the same goal.