Linear Algebra

Write your **name**: ________________________________ (2 points).

In **problems 1–5**, circle the correct answer. (5 points per problem)

1. The equation $L(A) = A - A^T$ defines a linear operator L on the vector space of $n \times n$ matrices. `True` `False`

   Solution. True. Taking the transpose is a linear operation, and so is subtraction. Explicitly, if A_1 and A_2 are two matrices, and c_1 and c_2 are scalars, then

$$\begin{aligned} L(c_1A_1 + c_2A_2) &= (c_1A_1 + c_2A_2) - (c_1A_1 + c_2A_2)^T \\ &= (c_1A_1 + c_2A_2) - (c_1A_1^T + c_2A_2^T) \\ &= c_1(A_1 - A_1^T) + c_2(A_2 - A_2^T) \\ &= c_1L(A_1) + c_2L(A_2). \end{aligned}$$

   Thus the transformation L does preserve linear combinations.

2. A 3×5 matrix B always has the same rank as the 5×3 matrix B^T. `True` `False`

   Solution. True. The rank of B equals the dimension of the row space of B. An important theorem says that the rank of B also equals the dimension of the column space of B. But the column space of B is the row space of B^T, so the rank of B equals the rank of B^T.

3. If the linear system $A\mathbf{x} = \mathbf{b}$ is consistent, then the vector $\mathbf{b}$ must belong to the null space of A^T. `True` `False`

   Solution. False. The system is consistent if and only if the vector $\mathbf{b}$ belongs to $R(A)$, the column space of A. This space equals the *orthogonal complement* of the null space of A^T.

   Example: If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then the linear system $A\mathbf{x} = \mathbf{b}$ is consistent, and the null space of A^T is the span of the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, so $\mathbf{b}$ does not belong to $N(A^T)$.

4. The transition matrix corresponding to a change of basis in R^n must be an invertible matrix. `True` `False`

   Solution. True. The inverse matrix corresponds to the inverse change of basis.

5. If the matrix representing a linear transformation $L\colon R^n \to R^n$ with respect to the standard basis has a row of zeroes, then one of the standard basis vectors belongs to the kernel of L. `True` `False`

   Solution. False. One of the standard basis vectors belongs to the kernel of L if the matrix has a *column* of zeroes.

   Example: The matrix $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ has a row of zeroes, and the two standard basis vectors both have image equal to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so neither basis vector is in the kernel of the linear transformation. The kernel of the linear transformation is the span of the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

In **problems 6–9**, fill in the blanks. (7 points per problem)

6. If A is a 30×4 matrix of rank 4, then $\dim N(A)$, the dimension of the null space of A, equals __________ .

   Solution. According to the rank–nullity theorem, the dimension of the null space of A equals $4 - 4$, or 0.

7. If L is the linear operator on R^2 that first reflects in the x-axis and then rotates by 45° counterclockwise, then the matrix representation of L (with respect to the standard basis) is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \square \\ \square & \square \end{pmatrix}$$

Solution. The first standard basis vector $\begin{pmatrix}1\\0\end{pmatrix}$ reflects to itself and then rotates to $\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$, which is the first column of the matrix. The second standard basis vector $\begin{pmatrix}0\\1\end{pmatrix}$ reflects to $\begin{pmatrix}0\\-1\end{pmatrix}$ and then rotates to $\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$, which is the second column of the matrix. Thus the matrix representation of L is $\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$.

8. If A and B are $n \times n$ matrices, and there exists a nonsingular matrix S such that $B = S^{-1}AS$, then the matrices A and B are called

   ____________________ .

   Solution. Such matrices A and B are *similar* matrices.

9. In R^2, the vector projection of $\begin{pmatrix}-1\\2\end{pmatrix}$ onto $\begin{pmatrix}3\\4\end{pmatrix}$ is the vector $\begin{pmatrix}\square\\\square\end{pmatrix}$.

   Solution. A unit vector in the direction of $\begin{pmatrix}3\\4\end{pmatrix}$ is $\begin{pmatrix}3/5\\4/5\end{pmatrix}$. The required vector projection is this unit vector multiplied by its scalar product with the vector $\begin{pmatrix}-1\\2\end{pmatrix}$: namely,

$$\left\langle \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}3/5\\4/5\end{pmatrix} \right\rangle \begin{pmatrix}3/5\\4/5\end{pmatrix} = \begin{pmatrix}3/5\\4/5\end{pmatrix}.$$

In **problems 10–12**, show your work and explain your method.
(15 points per problem)

10. Suppose $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Determine a basis for $R(A^T)$, the range of the transpose of A.

    Solution. The range of A^T is the column space of A^T, which is the same as the row space of A. Since the two rows of A evidently are linearly independent (they are not multiples of each other), the two rows of A already form a basis for the row space of A. Thus one basis for $R(A^T)$ is the pair of vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.

    Alternatively, you could compute the reduced row echelon form of A, which is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$. The vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ form a somewhat simpler basis for $R(A^T)$.

11. Find the distance in the x–y plane from the point $(4, 1)$ to the line with equation $20x + 10y = 0$.

    Solution. The simplest method is to find the scalar projection of the vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ onto the direction *orthogonal* to the line. In other words, compute the scalar product of the vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ with a unit vector orthogonal to the line. One vector orthogonal to the line is $\begin{pmatrix} 20 \\ 10 \end{pmatrix}$; a simpler vector in the same direction is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; a unit vector in the same direction is $\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$. Therefore the distance from the point to the line equals

$$\left\langle \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \right\rangle = 9/\sqrt{5}.$$

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12. Suppose $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, and let $\mathbf{e}_1$ and $\mathbf{e}_2$ denote the two standard basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Suppose that a linear transformation $L\colon R^2 \to R^2$ is represented with respect to the basis $[\mathbf{u}_1, \mathbf{u}_2]$ by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Find the matrix representation of L with respect to the standard basis $[\mathbf{e}_1, \mathbf{e}_2]$.

Solution. If $S = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}$, then S is the transition matrix from the **u**-basis to the standard basis. Therefore the required matrix representation is

$$S \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} S^{-1}.$$

Now $\det(S) = -1$, so $S^{-1} = \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}$. Consequently, the matrix representation of L with respect to the standard basis equals

$$\begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix},$$

which works out to be the matrix $\begin{pmatrix} 5 & -1 \\ 12 & -2 \end{pmatrix}$.