Linear Algebra

Write your **name**: ______________________________ (2 points).

In **problems 1–5**, circle the correct answer. (5 points per problem)

1. Every linearly independent set of three vectors in R^3 is a basis for R^3.
   `True` `False`

   Solution. True. Three linearly independent vectors span a three-dimensional space, hence all of R^3. A linearly independent spanning set is a basis.

2. For $n \times n$ matrices A and B, the determinant of the product AB always equals the determinant of BA. `True` `False`

   Solution. True. The determinant is a multiplicative function: namely, $\det(AB) = \det(A)\det(B)$. Similarly, $\det(BA) = \det(B)\det(A)$. But $\det(B)\det(A) = \det(A)\det(B)$ because multiplication of *numbers* is commutative (although multiplication of matrices is not commutative).

3. Every orthogonal 3×3 matrix has rank 3. `True` `False`

   Solution. True. Being an orthonormal set, the columns of an orthogonal matrix are a linearly independent set. Hence the dimension of the column space (which equals the rank) equals 3.

4. In an inner product space, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. `True` `False`

   Solution. False. The indicated property holds if and only if the vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal*.

5. If L is a linear transformation mapping R^3 into R^2, then there is a 2×3 matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every vector $\mathbf{x}$ in R^3.
   `True` `False`

   Solution. True. This statement is a special case of the fundamental matrix representation of a linear transformation (Theorem 4.2.1 on page 185).

Linear Algebra

In **problems 6–9**, fill in the blanks. (7 points per problem)

6. If A is a square matrix, λ is a scalar, and $\mathbf{x}$ is a nonzero vector such that $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is called ____________________ .

   Solution. Such a vector is an eigenvector of the matrix A.

7. $\det \begin{pmatrix} 3 & 0 & 4 \\ 5 & 0 & 2 \\ 8 & \square & 6 \end{pmatrix} = 28.$

   Solution. A cofactor expansion on the middle column shows that the missing number x has the property that $-x \begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} = 28$, or $14x = 28$, or $x = 2$.

8. Vectors $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$ form an orthonormal basis for R^3.

   Solution. The two possible answers are $\pm \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$.

9. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, then $\dim N(A)$, the nullity of A, equals _____ .

   Solution. Since there is only one linearly independent row, the matrix has rank 1. By the rank–nullity theorem, the nullity equals 2.

Linear Algebra

In **problems 10–12**, show your work and explain your method.
(15 points per problem)

10. Suppose $A = \begin{pmatrix} 1 & -6 \\ 3 & 12 \end{pmatrix}$. Find a lower-triangular matrix L and an upper-triangular matrix U such that $A = LU$.

Solution. The elementary matrix that implements the row operation $R2 \to R2 - 3R1$ is $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$. In other words,

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -6 \\ 3 & 12 \end{pmatrix} = \begin{pmatrix} 1 & -6 \\ 0 & 30 \end{pmatrix}.$$

Multiplying both sides by the inverse elementary matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ shows that

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -6 \\ 0 & 30 \end{pmatrix}.$$

Thus $L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & -6 \\ 0 & 30 \end{pmatrix}$.

There are infinitely many other correct answers, for A can also be written as $(LD)(D^{-1}U)$ for an arbitrary invertible diagonal matrix D.

11. Find the least-squares solution to the following inconsistent system.

$$\begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 - x_2 &= 0 \\ x_1 - x_2 &= 2 \end{aligned}$$

Solution. In matrix form, the system reads as follows:

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}.$$

The corresponding least-squares problem is formed by multiplying both sides by $\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$, the transpose matrix. The resulting system has the following form:

$$\begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \end{pmatrix}.$$

In other words, $6x_1 = 12$ and $3x_2 = 3$, so $x_1 = 2$ and $x_2 = 1$.

12. Suppose $A = \begin{pmatrix} -2 & -2 \\ 15 & 9 \end{pmatrix}$. Find a diagonal matrix D and an invertible matrix S such that $S^{-1}AS = D$.

Solution. The diagonal matrix D should have the eigenvalues of A on the diagonal, and the matrix S should be the transition matrix that has the corresponding eigenvectors of A as its columns.

The characteristic equation for the eigenvalues is the following:

$$\begin{aligned} 0 &= \begin{vmatrix} -2-\lambda & -2 \\ 15 & 9-\lambda \end{vmatrix} = (-2-\lambda)(9-\lambda) - (-2) \times 15 \\ &= \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4). \end{aligned}$$

Therefore the eigenvalues are 3 and 4, and $D = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

An eigenvector corresponding to eigenvalue 3 is any nonzero vector in the nullspace of the matrix $\begin{pmatrix} -5 & -2 \\ 15 & 6 \end{pmatrix}$, and one such vector is $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$. An eigenvector corresponding to eigenvalue 4 is any nonzero vector in the nullspace of the matrix $\begin{pmatrix} -6 & -2 \\ 15 & 5 \end{pmatrix}$, and one such vector is $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$. Therefore one choice for S is $\begin{pmatrix} 2 & 2 \\ -5 & -6 \end{pmatrix}$.

Other correct answers are possible. You could multiply the first column of S by any nonzero number and multiply the second column of S by another nonzero number. Or you could swap the columns of S and simultaneously swap the diagonal entries of D.