1. To solve the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = -30\cos(2x)$, first solve the homogeneous equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$ by trying $y = e^{rx}$. This leads to the quadratic equation $r^2 - 5r + 4 = 0$, or $(r-4)(r-1) = 0$. Hence the general solution of the homogeneous equation is $y = c_1e^x + c_2e^{4x}$, where c_1 and c_2 are arbitrary constants.

   Next look for a particular solution of the original differential equation. Using the method of undetermined coefficients, try $y = A\cos(2x) + B\sin(2x)$, where the constants A and B are to be determined. Substituting this trial solution into the differential equation and matching coefficients of $\cos(2x)$ and $\sin(2x)$ on both sides gives a pair of simultaneous equations $-4A - 10B + 4A = -30$ and $-4B + 10A + 4B = 0$, whence $A = 0$ and $B = 3$.

   The general solution of the original differential equation is therefore $y = c_1e^x + c_2e^{4x} + 3\sin(2x)$. Applying the initial conditions $y(0) = 5$ and $y'(0) = 26$ gives a pair of simultaneous equations $5 = c_1 + c_2$ and $26 = c_1 + 4c_2 + 6$, whence $c_1 = 0$ and $c_2 = 5$.

   The final solution is then $y = 5e^{4x} + 3\sin(2x)$.

2. This problem is the same as exercise 35 on page 229 of Nagle & Saff. It was an assigned homework problem!

To solve the differential equation $\frac{d^2y}{dx^2} + y = \sec(x)$, first observe that the homogeneous equation $y'' + y = 0$ has the general solution $c_1\cos(x) + c_2\sin(x)$. Using the method of variation of parameters, look for a solution of the nonhomogeneous equation in the form $y = v_1\cos(x) + v_2\sin(x)$, where v_1 and v_2 are functions to be determined.

If we impose the side condition $v_1'\cos(x) + v_2'\sin(x) = 0$, then the derivative y' has the simple form $y' = -v_1\sin(x) + v_2\cos(x)$. Then $y'' = -v_1\cos(x) - v_2\sin(x) - v_1'\sin(x) + v_2'\cos(x)$. Substituting this information into the differential equation yields $-v_1'\sin(x) + v_2'\cos(x) = \sec(x)$.

The side condition and this equation form a pair of simultaneous equations for v_1' and v_2'. One way to solve is to multiply the first equation by $\sin(x)$, the second equation by $\cos(x)$, and add the resulting equations to get $v_2' = 1$ (since $\sin^2(x) + \cos^2(x) = 1$ and $\cos(x)\sec(x) = 1$). Consequently, we can take $v_2 = x$. Inserting $v_2' = 1$ in the side condition gives $v_1' = -\sin(x)/\cos(x)$, and integrating with the substitution $u = \cos(x)$ yields $v_1 = \ln(\cos(x))$.

The general solution to the original differential equation is then $y = c_1\cos(x) + c_2\sin(x) + \{\ln(\cos(x))\}\cos(x) + x\sin(x)$.

3. To set up the system of differential equations, use that the force exerted by a spring is proportional to the stretch in

the spring (Hooke's law). The stretch in the spring joining the two masses depends on the positions of *both* masses.

Let x denote the displacement of the heavier mass from its equilibrium position, and let y denote the displacement of the lighter mass from its equilibrium position. Suppose that x and y increase to the right, and the lighter mass is to the right of the heavier mass. From Newton's law ($F = ma$) and from Hooke's law we get the differential equations $2 \cdot x'' = -2 \cdot x + 1 \cdot (y - x)$ and $1 \cdot y'' = -1 \cdot (y - x)$, where the primes represent derivatives with respect to time t.

One way to solve the pair of differential equations is to isolate y in the first equation: $y = 2x'' + 3x$. Substituting this relation into the second equation gives $2x^{iv} + 3x'' = -(2x'' + 3x - x)$, or $2x^{iv} + 5x'' + 2x = 0$. Trying $x = e^{rt}$ gives $2r^4 + 5r^2 + 2 = 0$. The quadratic formula gives $r^2 = (-5 \pm \sqrt{25 - 16})/4$, whence r^2 is either -2 or $-1/2$. Therefore r is either $\pm\sqrt{2}\,i$ or $\pm\sqrt{1/2}\,i$. Consequently, the natural frequencies of the system are $\sqrt{2}/2\pi$ and $\sqrt{1/2}/2\pi$.

4. There are many different RLC circuits for which $I(t) = (5 + 4t)e^{-3t}$ will be a solution of the differential equation
$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}.$$
For example, if $E(t)$ is constant, then $dE/dt = 0$, and the differential equation is homogeneous. To have a solution of the specified form, the system must be critically damped; in other words, the quadratic equation $Lr^2 + Rr + 1/C = 0$ must have the double root $r = -3$. Since $(r+3)^2 = r^2+6r+9$, we could take $L = 1$, $R = 6$, and $C = 1/9$.

   Other examples of valid circuits can be obtained under the assumption that $dE/dt \neq 0$. In that case, an acceptable E can be computed for practically any values of L, R, and C. A simple specific case would be $L = 0$, $R = 0$, and $C = 1$. The differential equation then reduces to $I = dE/dt$, and since $I(t)$ is prescribed, we can integrate to find $E(t) = K - (19 + 12t)e^{-3t}/9$, where K is an arbitrary integration constant. As another example, choosing $L = 1$, $R = 1$, and $C = 1$ and integrating gives $E(t) = K - (84t+73)e^{-3t}/9$. The value of the constant K can be adjusted to make the voltage positive, if desired.

5. Substituting $y = uv$ in the differential equation $y'' + py' + qy = 0$ and simplifying yields $u''v + u'(2v' + pv) + u(v'' + pv' + qv) = 0$. To make the u' term vanish, we need $2v' + pv = 0$. This is a first-order linear differential equation for v. Separating variables gives $v'/v = -p/2$, and integrating yields $\ln v = -\int \frac{1}{2}p(x)\,dx$. Hence $v = \exp(-\frac{1}{2}\int p(x)\,dx)$.