Math 311-102

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Parametric curves in space

Example. The vector function $f(t) = (\cos(t), \sin(t), t)$ represents a curve in $\mathbb{R}^3$ (a helix). Find an equation for the line tangent to the curve at the point where $t = \pi/2$.

Solution. The derivative $f'(t)$ gives the velocity vector (= tangent vector): $(-\sin(t), \cos(t), 1)\big|_{t=\pi/2} = (-1, 0, 1)$. The point on the curve corresponding to $t = \pi/2$ is $(0, 1, \pi/2)$. A parametric equation for the tangent line in terms of parameter s is $(0, 1, \pi/2) + s(-1, 0, 1)$.

Continuation. What is the arc length of the curve from $t = 0$ to $t = 2\pi$?

Solution. To get the arc length, integrate the speed (length of the velocity vector): $\int_0^{2\pi} |f'(t)|\,dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\,dt$ $= \int_0^{2\pi} \sqrt{2}\,dt = 2\pi\sqrt{2}$.

Parametric surfaces

Example. As u and v vary from 0 to 2π, the vector function $f(u, v) = ((5 + 2\sin u)\cos v, (5 + 2\sin u)\sin v, 2\cos u)$ sweeps out a surface in $\mathbb{R}^3$ (a torus). Find an equation for the plane tangent to the surface at the point where $u = \pi/6$ and $v = \pi/2$.

Solution. The point on the surface is $(0, 6, \sqrt{3})$. The partial derivative $\frac{\partial f}{\partial u}$ is a vector tangent to the surface: $((2\cos u)\cos v, (2\cos u)\sin v, -2\sin u)\big|_{\substack{u=\pi/6\\ v=\pi/2}} = (0, \sqrt{3}, -1)$.

Similarly, $\frac{\partial f}{\partial v}$ is a tangent vector: $(-(5 + 2\sin u)\sin v, (5 + 2\sin u)\cos v, 0)\big|_{\substack{u=\pi/6\\ v=\pi/2}} = (-6, 0, 0)$.

The cross product of the two tangent vectors gives a vector normal to the surface: $(0, 6, 6\sqrt{3})$. Just as good a normal is the scalar multiple $(0, 1, \sqrt{3})$. The tangent plane has equation $0(x - 0) + 1(y - 6) + \sqrt{3}(z - \sqrt{3}) = 0$ or $y + \sqrt{3}z = 9$.

Directional derivatives

Example. If $f(x, y, z) = x^2y + xe^z$, find the rate of change of f in the direction of the unit vector $(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7})$ at the point $(2, 1, 0)$.

Solution. Each of the three partial derivatives of f contributes, and the rate of change equals the sum $\left(\frac{2}{7}\frac{\partial f}{\partial x} - \frac{3}{7}\frac{\partial f}{\partial y} + \frac{6}{7}\frac{\partial f}{\partial z}\right)\big|_{(2,1,0)}$ $= \left(\frac{2}{7}(2xy + e^z) - \frac{3}{7}x^2 + \frac{6}{7}xe^z\right)\big|_{(2,1,0)} = \frac{10}{7}$.

Notation. The *gradient* of f, written ∇f, means the vector $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ of partial derivatives. The directional derivative of f in the direction of the unit vector $\vec{u}$ equals the dot product $(\nabla f)\cdot\vec{u}$. The length $|\nabla f|$ represents the largest rate of change of f in any direction.