Math 311-102

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One-variable chain rule

Example. You learned in your first calculus course that $\frac{d}{dx}\ln(\sin(x)) = \frac{1}{\sin(x)}\cos(x)$.

The general rule for differentiating a composite function $f \circ g$ is $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$.

Alternate notation. If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

Example. If you forgot the formula for $\frac{d}{dx}\sin^{-1}(x)$, how could you work it out?

Solution. Put $y = \sin^{-1}(x)$, which implies that $\sin(y) = x$. We want to find $\frac{dy}{dx}$, and the chain rule implies that this equals $\frac{1}{dx/dy} = \frac{1}{\cos(y)}$. Since $\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$, it follows that $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$.

Linear functions

Example. If $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, what is the derivative matrix (Jacobian matrix) of f?

Answer. The derivative matrix has rows equal to the gradients of the component functions of f, so the Jacobian matrix is $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

In other words, a linear function is its own linear approximation.

You compose linear functions by multiplying their matrices.

Therefore the derivative matrix of a (multi-variable) composite function equals the product of the derivative matrices.

Example: multi-variable chain rule

If $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x^2 + y \\ xy \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = g\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$ (polar coordinates), find the derivative of the composite function $f \circ g$.

Solution. The derivative matrix of f (with respect to x and y) equals $\begin{pmatrix} 1 & 1 \\ 2x & 1 \\ y & x \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2r\cos(\theta) & 1 \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}$, and the derivative matrix of g equals $\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}$; the product equals

$$\begin{pmatrix} \cos(\theta) + \sin(\theta) & -r\sin(\theta) + r\cos(\theta) \\ 2r\cos^2(\theta) + \sin(\theta) & -2r^2\cos(\theta)\sin(\theta) + r\cos(\theta) \\ 2r\sin(\theta)\cos(\theta) & -r^2\sin^2(\theta) + r^2\cos^2(\theta) \end{pmatrix}.$$

Example continued

Alternate notation. If $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x+y \\ x^2+y \\ xy \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$, then

$$\begin{pmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}.$$

For example, $\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$.

Coordinate transformations

Question. Is the polar coordinate transformation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$ invertible?

Answer. You can solve explicitly for $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(\frac{y}{x}) = \cot^{-1}\frac{x}{y}$, but there are two problems. There is a *global* problem that θ is ambiguous by addition of multiples of 2π. There is a *local* problem that θ is not defined when $x = y = 0$. The local problem is addressed by the

Inverse function theorem. A transformation is locally invertible at points where the Jacobian matrix is invertible.

Example. Since $\det\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r$, the theorem confirms that the polar coordinate transformation is locally invertible when $r \neq 0$.