Math 311-102

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Recap: Green's theorem in the plane

$$\int_{\substack{\text{closed}\\\text{curve}}} P(x,y)\,dx + Q(x,y)\,dy = \iint_{\substack{\text{region}\\\text{inside}}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

or, equivalently, $\displaystyle\int_{\substack{\text{closed}\\\text{curve}}} \vec{F}\cdot\vec{t}\,ds = \iint_{\substack{\text{region}\\\text{inside}}} (\nabla\times\vec{F})\cdot\vec{k}\,dx\,dy$

($\vec{t}$ is the unit tangent vector and ds is the arclength element)

or, with a different choice of the vector field $\vec{F}$,

$$\int_{\substack{\text{closed}\\\text{curve}}} \vec{F}\cdot\vec{n}\,ds = \iint_{\substack{\text{region}\\\text{inside}}} (\nabla\cdot\vec{F})\,dx\,dy$$

($\vec{n}$ is the unit normal vector).

Conservative fields

A vector field $\vec{F}$ is called *conservative* if
(a) the path integral $\int_C \vec{F}\cdot d\vec{x}$ depends only on the endpoints of the curve C, but not on the details of the path,
or, equivalently,
(b) the path integral $\oint_C \vec{F}\cdot d\vec{x}$ equals 0 for every *closed* path C,
or, equivalently,
(c) the vector field $\vec{F}$ is a gradient field: $\vec{F} = \nabla f$ for some function f (called a *potential* function).

Example. If $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$, evaluate $\int_C \vec{F}\cdot\vec{t}\,ds$ over any path C joining the points $(0,0,0)$ and $(1,2,3)$.

Solution. If $\vec{F} = \nabla f$, then $\int_C \nabla f\cdot\vec{t}\,ds = \int_C \nabla f\cdot d\vec{x}$ $= f(1,2,3) - f(0,0,0)$. By inspection, $f(x,y,z) = xyz$, so the answer is $1\times 2\times 3 - 0\times 0\times 0 = 6$.

Finding potential functions

Example. If $\vec{F}(x,y,z) = (2x\sin z, 2ye^z, x^2\cos z + y^2e^z + 5)$, find a potential function $f(x,y,z)$ such that $\vec{F} = \nabla f$.

Solution. First of all, we want $\frac{\partial f}{\partial x} = 2x\sin z$. Integrate with respect to x to get $f(x,y,z) = x^2\sin z + g(y,z)$ for some function g.

Then $2ye^z = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$. Therefore $g(y,z) = y^2e^z + h(z)$ for some function h, and so $f(x,y,z) = x^2\sin z + y^2e^z + h(z)$.

Now $x^2\cos z + y^2e^z + 5 = \frac{\partial f}{\partial z} = x^2\cos z + y^2e^z + h'(z)$, so $h(z) = 5z + c$ for any constant c.

Thus $f(x,y,z) = x^2\sin z + y^2e^z + 5z + c$.

When do potentials exist?

Example. If $\vec{F}(x,y) = (y,-x)$, then $\vec{F}$ is not a gradient field.

In fact, if $\vec{F}(x,y) = \nabla f(x,y)$, then

$\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = -x$. Consequently,

$\frac{\partial^2 f}{\partial y \partial x} = 1$ and $\frac{\partial^2 f}{\partial x \partial y} = -1$. Impossible.

A *necessary* condition for a vector field $\vec{F} = (F_1, F_2, \ldots, F_n)$ (having continuous derivatives) to be a gradient field is that $\frac{\partial F_j}{\partial x_k} = \frac{\partial F_k}{\partial x_j}$ for all j and k. This says that the Jacobian matrix of $\vec{F}$ is a *symmetric* matrix. If $n = 3$, this says that $\nabla \times \vec{F} = 0$.

The above condition is also *sufficient* when the domain of $\vec{F}$ is a *simply connected* region.