

# Linear Algebra

**Instructions** Please answer the six problems on your own paper. These are essay questions: you should write in complete sentences.

1. Are the two matrices

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ 5 & 2 & 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 4 & 4 & -2 \\ 5 & 5 & 2 \end{bmatrix}$$

row equivalent? Explain why or why not.

[Recall that two matrices are row equivalent if one matrix can be converted into the other by a sequence of elementary row operations.]

2. Determine all values of  $t$  for which the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3t & 4 & 0 & 0 \\ 0 & 5t & 6 & 0 \\ 7 & 0 & 0 & 8t \end{bmatrix}$$

is singular (that is, not invertible).

3. Create a  $3 \times 3$  matrix from your nine-digit student identification number by putting the first three digits in the first row of the matrix, the next three digits in the second row of the matrix, and the final three digits in the third row of the matrix. Determine the dimension of the column space of your matrix. Explain your method.
4. In the vector space  $C[0, 1]$  of continuous functions, the set of functions  $[1, x, \sin(x), \cos(x)]$  forms a basis  $B$  for a certain four-dimensional subspace  $W$ . Find the  $4 \times 4$  matrix that represents, with respect to the basis  $B$ , the linear operator  $D$  on  $W$  defined by differentiation. [That is,  $D(1) = 0$ ,  $D(x) = 1$ ,  $D(\sin(x)) = \cos(x)$ , and  $D(\cos(x)) = -\sin(x)$ .] Explain your reasoning.
5. Find the line of the form  $y = c_1 + c_2x$  that gives the best least-squares fit to the following data:

$$\begin{array}{c|c|c|c} x & 0 & 1 & 2 \\ \hline y & 1 & 3 & -1 \end{array}$$

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6. Find a diagonal matrix that is similar to the matrix

$$\begin{bmatrix} 3 & 2 & 3 \\ 5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

[Recall that two square matrices are similar if they represent the same linear operator with respect to two different bases. Equivalently,  $A$  and  $B$  are similar matrices if  $B = S^{-1}AS$  for some invertible matrix  $S$ .] Explain your method.

**Bonus (up to 10 points extra credit)**

The so-called *Legendre polynomials* are an orthogonal set of functions with respect to the inner product

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x) dx.$$

The Legendre polynomial  $P_n(x)$  is a polynomial of degree  $n$  that is normalized not by the condition  $\|P_n\| = 1$  but instead by the condition that  $P_n(1) = 1$ . Here are the first three Legendre polynomials:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1). \end{aligned}$$

Determine the degree-three polynomial  $P_3(x)$ .