

## Math 323

### Linear Algebra

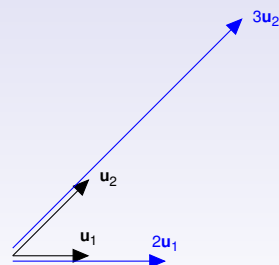
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## Example of eigenvectors

The matrix  $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$  defines a linear transformation of  $\mathbb{R}^2$  that is easy to understand. The transformation stretches the vector  $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  by a factor of 2 and stretches the vector  $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  by a factor of 3.



The vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are called *eigenvectors*, and the scale factors 2 and 3 are the corresponding *eigenvalues*.

The transformation is particularly simple to describe in the basis  $[\mathbf{u}_1, \mathbf{u}_2]$ : namely, the matrix  $U^{-1}AU$  is the diagonal matrix  $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ .

## Computing eigenvectors

**Example.** The matrix  $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$  has 3 as one of its eigenvalues. Find a corresponding eigenvector.

**Solution.** We seek a vector  $\mathbf{v}$  such that  $A\mathbf{v} = 3\mathbf{v}$ . Equivalently,  $\mathbf{v}$  should be in the nullspace of the matrix  $A - 3I$  ( $I =$  identity). Find the nullspace by row reduction:

$$\begin{array}{l} \left( \begin{array}{ccc|c} 9 & 4 & -5 & 0 \\ -8 & -3 & 5 & 0 \\ 10 & 4 & -6 & 0 \end{array} \right) \xrightarrow{R_1 \rightarrow R_1 + R_2} \left( \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -8 & -3 & 5 & 0 \\ 10 & 4 & -6 & 0 \end{array} \right) \\ \xrightarrow{\substack{R_2 \rightarrow R_2 + 8R_1 \\ R_3 \rightarrow R_3 - 10R_1}} \left( \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & -6 & -6 & 0 \end{array} \right) \xrightarrow{\text{three steps}} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right). \end{array}$$

Then  $\mathbf{v} = (1, -1, 1)^T$  is an eigenvector with eigenvalue 3.

## Computing eigenvalues (continued)

**Example.** The matrix  $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$  has other eigenvalues besides the number 3. Find them.

**Solution.** The condition for a number  $\lambda$  to be an eigenvalue of  $A$  is that the matrix  $A - \lambda I$  has a non-trivial nullspace. Equivalently,  $\det(A - \lambda I) = 0$ , the *characteristic equation*:

$$\begin{aligned} 0 &= \begin{vmatrix} 12 - \lambda & 4 & -5 \\ -8 & 0 - \lambda & 5 \\ 10 & 4 & -3 - \lambda \end{vmatrix} \xrightarrow[\substack{R_1 \rightarrow \\ R_1 + R_2}]{C_2 \rightarrow} \begin{vmatrix} 4 - \lambda & 4 - \lambda & 0 \\ -8 & -\lambda & 5 \\ 10 & 4 & -3 - \lambda \end{vmatrix} \\ &\xrightarrow{C_2 - C_1} (4 - \lambda) \begin{vmatrix} 4 - \lambda & 0 & 0 \\ -8 & 8 - \lambda & 5 \\ 10 & -6 & -3 - \lambda \end{vmatrix} = (4 - \lambda) \begin{vmatrix} 8 - \lambda & 5 \\ -6 & -3 - \lambda \end{vmatrix} \\ &= (4 - \lambda)(\lambda^2 - 5\lambda + 6) = (4 - \lambda)(\lambda - 3)(\lambda - 2). \end{aligned}$$

Therefore the eigenvalues of  $A$  are 4, 3, and 2.

## Exercise

If  $A = \begin{pmatrix} 12 & 4 & -5 \\ -8 & 0 & 5 \\ 10 & 4 & -3 \end{pmatrix}$ , find eigenvectors corresponding to the eigenvalues 2 and 4.

We already know that  $(1, -1, 1)^T$  is an eigenvector with eigenvalue 3.

**Answer.** Vector  $(4, -3, 4)^T$  is an eigenvector with eigenvalue 4, and  $(5, -5, 6)^T$  is an eigenvector with eigenvalue 2.

**Remark.**  $S = \begin{pmatrix} 4 & 1 & 5 \\ -3 & -1 & -5 \\ 4 & 1 & 6 \end{pmatrix}$  is the transition matrix from an eigenvector basis to the standard basis, and  $S^{-1}AS$  is a diagonal matrix with the eigenvalues 4, 3, and 2 on the diagonal.