

Linear Algebra

Instructions Please answer the five problems on your own paper. These are essay questions: you should write in complete sentences.

1. Jordan is using a TI-89 calculator to help analyze the linear system $A\mathbf{x} = \mathbf{b}$, where A is a certain 3×4 matrix and \mathbf{b} is a certain 3×1 matrix (a column vector). Jordan applies the `rref` command to the augmented coefficient matrix and obtains the result

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

(Jordan's calculator does not show a vertical bar to separate the last column from the coefficient matrix.) Discuss what information Jordan can deduce about the original linear system.

[For instance, is the system underdetermined or overdetermined? consistent or inconsistent? Is there a unique solution? Does Jordan have enough information to write down the solution(s)?]

Solution. The indicated reduced row echelon form reveals that we have a consistent, underdetermined system with lead variables x_1 and x_3 and free variables x_2 and x_4 . There are infinitely many solutions, and we can exhibit the solutions as follows:

$$\mathbf{x} = \begin{bmatrix} 4 - 2\beta \\ \alpha \\ -5 - 3\beta \\ \beta \end{bmatrix}, \quad \text{where } \alpha \text{ and } \beta \text{ take arbitrary values.}$$

2. Consider the system of three simultaneous equations

$$\begin{cases} x_1 + x_2 & = 2 \\ ax_1 + ax_2 & = 3a \\ bx_1 + bx_2 + ax_3 & = 4 + b \end{cases}$$

for the unknowns x_1 , x_2 , and x_3 . Find all values of a and b for which this system of equations is consistent.

Explain your reasoning in complete sentences.

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Solution. Method 1 (using rows). The most popular approach was to bring the augmented coefficient matrix to row echelon form as follows:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ a & a & 0 & 3a \\ b & b & a & 4+b \end{array} \right] & \xrightarrow[\substack{R2 \rightarrow R2 - aR1 \\ R3 \rightarrow R3 - bR1}]{} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 4-b \end{array} \right] \\ & \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & a & 4-b \\ 0 & 0 & 0 & a \end{array} \right]. \end{aligned}$$

The bottom row of this echelon form shows that a necessary condition for the system to be consistent is that $a = 0$. Inserting this condition into the second row shows that another necessary condition for consistency is that $b = 4$.

When both of these necessary conditions ($a = 0$ and $b = 4$) hold, the echelon form reduces to

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

which evidently represents a consistent system. Indeed, we can exhibit the solutions as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - \alpha \\ \alpha \\ \beta \end{bmatrix}, \quad \text{where } \alpha \text{ and } \beta \text{ take arbitrary values.}$$

Thus the two conditions $a = 0$ and $b = 4$ together are necessary and sufficient for the system to be consistent.

Method 2 (using columns). By the consistency theorem for linear systems (Theorem 1.3.1), we know that the system is consistent if the column vector on the right-hand side is a linear combination of the columns of the coefficient matrix. The first two columns of the coefficient matrix are identical, so we want to know if there exist constants c_1 and c_2 for which

$$c_1 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} 2 \\ 3a \\ 4+b \end{bmatrix}.$$

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Looking at the first component of the column vectors, we see that the only possible choice for c_1 is 2. Then looking at the second component of the column vectors, we find that $2a = 3a$, which means that $a = 0$. Looking now at the third component of the column vectors, we see that $2b = 4 + b$, which means that $b = 4$. In conclusion, we can write the column vector on the right-hand side as a linear combination of the columns of the coefficient matrix if and only if both $a = 0$ and $b = 4$.

Remark The TI-89 calculator will answer this problem incorrectly, unless you are very careful about what question you ask it. If you apply the `rref` command to the augmented coefficient matrix, then the calculator returns a matrix with bottom row $[0\ 0\ 0\ 1]$, falsely indicating that the system is always inconsistent. (The calculator silently divides by a , ignoring the special case when $a = 0$.)

3. Suppose

$$A = \begin{bmatrix} 0 & a & 1 \\ 1 & 0 & 1 \\ 0 & 0 & a \end{bmatrix}.$$

Determine the value(s) of a for which the matrix A is invertible.

[If you do a computation to solve this problem, say what computation you are doing and why.]

Solution. We know that a square matrix is invertible precisely when its determinant is non-zero. Therefore we can answer the question by computing $\det(A)$. A cofactor expansion using the bottom row shows that

$$\det(A) = a \det \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} = -a^2.$$

Consequently, the matrix A is invertible precisely when $a \neq 0$.

4. Suppose that A is an $n \times n$ matrix, and S is an invertible $n \times n$ matrix. Show that $\det(S^{-1}AS) = \det(A)$.

Solution. We know that the determinant is a multiplicative function: the determinant of a product equals the product of the determinants. (See Theorem 2.2.3.) Therefore

$$\det(S^{-1}AS) = \det(S^{-1}) \det(A) \det(S).$$

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We also know that the product $\det(S^{-1})\det(S) = 1$. (This property was a homework problem: exercise 6 on page 104. The proof is the following calculation using the multiplicative property: $1 = \det(I) = \det(S^{-1}S) = \det(S^{-1})\det(S)$.) Therefore the right-hand side of the displayed equation simplifies to $\det(A)$, which is the required result.

There is a subtle point here. Matrix multiplication is not commutative, so we cannot simplify $S^{-1}AS$ to get A . But multiplication of *numbers* is commutative, so we *can* simplify $\det(S)^{-1}\det(A)\det(S)$ to get $\det(A)$.

5. Maude is studying the set of all polynomials in x of odd degree. Help Maude decide if this *set* forms a *vector space* (under the usual operations of addition and scalar multiplication).

Solution. Since the operations are the usual ones, the validity of the commutative, associative, and distributive laws is not in question. The set fails to be a vector space, however, for two reasons.

(1) There is no additive identity element in the set. The only candidate for the additive identity is the zero polynomial, which is not a polynomial of odd degree.

(2) The addition operation is not well defined; that is, the sum of two elements of the set need not be in the set. For instance, the polynomial $x^3 + x^2$ has odd degree (namely, degree 3), and the polynomial $-x^3 + x$ has odd degree, but their sum $x^2 + x$ has even degree and hence is not an element of the set. (We did an example like this in class.) In other words, our set is not *closed* under the addition operation.