Linear Algebra

Instructions Please answer the five problems on your own paper. These are essay questions: you should write in complete sentences.

1. Recall that P_3 denotes the vector space of polynomials of degree less than 3. Let S denote the two-dimensional subspace of P_3 consisting of polynomials $p(x)$ such that $p(0) = p(1)$. Find a basis for S, and explain how you know that your answer is a basis.

   Solution. A polynomial in P_3 has the form $ax^2 + bx + c$ for certain constants a, b, and c. Such a polynomial belongs to the subspace S if $a0^2 + b0 + c = a1^2 + b1 + c$, or $c = a + b + c$, or $0 = a + b$, or $b = -a$. Thus the polynomials in the subspace S have the form $a(x^2 - x) + c$. In other words, the two polynomials $x^2 - x$ and 1 are a spanning set for S. These two polynomials are also a linearly independent set since neither of these two polynomials is a scalar multiple of the other. Therefore the set $\{x^2 - x, 1\}$ is a basis for the subspace S.

2. Recall that $R^{2\times 2}$ denotes the vector space of all 2×2 matrices with real entries.

   (a) Show that the set of all *symmetric* 2×2 matrices with real entries is a subspace of $R^{2\times 2}$. (Recall that a matrix A is symmetric if $A = A^T$.)

   (b) What is the dimension of this subspace? How do you know?

   Solution. This problem is essentially problem 6 from Chapter Test B for Chapter 3 on page 173 of the textbook.

   (a) A symmetric 2×2 matrix has the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ for certain scalars a, b, and c. Evidently the sum of two matrices of this form is another matrix of the same type, and a scalar multiple of such a matrix is another matrix of the same type. Thus the set of symmetric matrices is closed under addition and closed under scalar multiplication, so the symmetric matrices do form a subspace of the space of 2×2 matrices.

(b) Since $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we see that the three symmetric matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ form a spanning set for the subspace of all symmetric matrices. Moreover, these three matrices are a linearly independent set since each has a 1 where the other two have 0's. Thus we have identified a basis for the subspace of symmetric matrices. This basis consists of three elements, so the dimension of our subspace equals three.

3. Give an example of a linear transformation $L\colon R^2 \to R^2$ whose kernel equals the span of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (There are many correct answers.)

Solution. There are infinitely many correct solutions. Two examples written as formulas are

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ 0 \end{pmatrix} \qquad \text{and} \qquad L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ 2x_2 - 2x_1 \end{pmatrix}.$$

The same two examples written in terms of matrix multiplication are

$$L(\mathbf{x}) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\mathbf{x} \qquad \text{and} \qquad L(\mathbf{x}) = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\mathbf{x}.$$

4. Suppose that $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. If $L\colon R^2 \to R^2$ is a linear transformation such that $L(\mathbf{u}_1) = \mathbf{u}_1$ and $L(\mathbf{u}_2) = 2\mathbf{u_2}$, find the matrix representation of L with respect to the *standard* basis.

Solution.

Method 1 (using similar matrices). The matrix representation of the transformation L with respect to the given nonstandard basis is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. The transition matrix from the nonstandard basis to the standard basis is $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. The transition matrix from the standard basis to the nonstandard basis is the inverse matrix $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1}$, or $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. The matrix representation of L with respect to the standard basis is then

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

which works out to $\begin{pmatrix} -2 & 2 \\ -6 & 5 \end{pmatrix}$.

Check:

$$\begin{pmatrix} -2 & 2 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$
$$\begin{pmatrix} -2 & 2 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Method 2 (bare hands). We seek a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Multiplying out these relations produces four linear equations for the four unknowns a, b, c, and d. Solving the equations yields $a = -2$, $b = 2$, $c = -6$, and $d = 5$.

Method 3 (working directly in standard coordinates). Observe that $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2\mathbf{u}_1 - 3\mathbf{u}_2$, so

$$L(\mathbf{e}_1) = 2L(\mathbf{u}_1) - 3L(\mathbf{u}_2) = 2\mathbf{u}_1 - 6\mathbf{u}_2 = \begin{pmatrix} -2 \\ -6 \end{pmatrix}.$$

This vector is the first column of the matrix representation of L in the standard basis. Similarly, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -\mathbf{u}_1 + 2\mathbf{u}_2$, so

$$L(\mathbf{e}_2) = -L(\mathbf{u}_1) + 2L(\mathbf{u}_2) = -\mathbf{u}_1 + 4\mathbf{u}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

This vector is the second column of the matrix representation of L in the standard basis.

5. Rose and Colin are studying a certain 3×4 matrix A. They use a TI-89 calculator to find the following reduced row echelon forms for the matrix A and for the transpose A^T:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Use this information to say as much as you can about the null space, the row space, and the column space of the original matrix A.
[Can you determine the dimension of each subspace? Can you determine a basis for each subspace?]

Solution. From the reduced row echelon form of the matrix A, you can read off that the two vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ form a basis for the row space of A, which thus has dimension 2.

It follows that the column space of A also has dimension 2, and the first and third columns of the original matrix A form a basis for the column space of A. Although the original matrix A is not given, we can nonetheless exhibit a basis for the column space by observing that the column space of A equals the row space of the transpose matrix A^T. The reduced row echelon form for A^T reveals that the two vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ form a basis for the column space of A. (Notice that in this example, the first and third columns of the reduced row echelon form of A definitely do not form a basis for the column space of A.)

Either from the rank-nullity theorem or from observing the two free variables in the reduced row echelon form of A, we deduce that the null

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space of A has dimension 2. The two vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ form a basis for the null space.