Complex Variables

Instructions Solve any **seven** of the following eight problems. Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Explain why $\displaystyle\int_{|z|=1} \frac{\sin(z)}{z}\,dz = 0$.

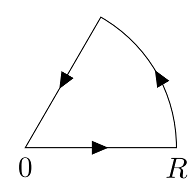
2. State **two** of the following four theorems:
   - Morera's theorem
   - Liouville's theorem
   - Rouché's theorem
   - Schwarz's lemma.

3. Give an example of a function that is analytic in the punctured plane (meaning the set $\{z : z \neq 0\}$) and that has a simple pole when $z = 0$, a double zero when $z = 1$, and no other zeroes or poles.

4. The function $\dfrac{1}{\sin(z)}$ has a Laurent series expansion in powers of z and z^{-1} valid when $0 < |z| < \pi$. Determine the first two nonzero terms of this expansion.

5. The function $\dfrac{\cos(z)}{z^3}$ has a pole of order 3 when $z = 0$. Determine the residue of this function at the pole.

6. The TI-89 calculator says that $\displaystyle\int_0^\infty \frac{1}{1+x^6}\,dx = \frac{\pi}{3}$. Prove this formula. Suggestion: integrate over a "piece of pie" of angle $\pi/3$.



7. The Fundamental Theorem of Algebra implies that the polynomial $3z^{28} - 2z^8 + 7z^5 + 1$ has 28 zeroes in the complex plane (counting multiplicities). How many of these 28 zeroes lie in the unit disc (the set where $|z| < 1$)? Explain how you know.

8. Student Max conjectures that if f and g are entire functions such that $|f(z)| \leq |g(z)|$ when $|z| = 1$, then $|f(z)| \leq |g(z)|$ when $|z| \leq 1$. If Max's conjecture is correct, then prove it; otherwise, supply a counterexample showing that Max is wrong.