Complex Variables

Instructions Please write your name in the upper right-hand corner of the page. Circle the correct answer. No explanation is required.

1. If z_1, z_2, and z_3 are three distinct complex numbers, then there is precisely one linear fractional transformation T such that $T(z_1) = 1$, $T(z_2) = i$, and $T(z_3) = 0$. `True` `False`

   Solution. True. It is a fundamental property of the set of linear fractional transformations that there is such a transformation taking three specified points to three specified points, and moreover this transformation is unique. See pages 198–199 in the textbook.

2. If f is an analytic function in a disc centered at z_0, and the derivative $f'(z_0) \neq 0$, then f is conformal at z_0. `True` `False`

   Solution. True. This statement is the local characterization of conformality. See page 210 in the textbook.

3. There exists a one-to-one conformal mapping from the open first quadrant onto the open unit disc. (The open first quadrant is the set $\{\, z : \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) > 0 \,\}$, and the open unit disc is $\{\, z : |z| < 1 \,\}$.)
   `True` `False`

   Solution. True. The Riemann mapping theorem guarantees the existence of such a map. Moreover, we can write down a formula: since z^2 opens out the first quadrant into the upper half plane, and the linear fractional map $\dfrac{z-i}{z+i}$ takes the upper half plane to the disc, the composite map $\dfrac{z^2-i}{z^2+i}$ does the trick. (The map does not, however, extend to be conformal on any domain containing 0.)

4. The two curves in the x–y plane defined by the equations $x^2 - y^2 = 1$ and $2xy = 3$ intersect orthogonally. `True` `False`

   Solution. True. You could solve this problem by using techniques from first-semester real calculus, but it is easier to observe that since $x^2 - y^2$ and $2xy$ are the real part and the imaginary part of the analytic function z^2, their level curves are orthogonal. (You did the general case for homework in exercise 15 on page 219 of the textbook.)

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5. The linear fractional transformation $\dfrac{z+2}{3z+1}$ maps the imaginary axis (together with the point at ∞) onto a circle whose radius equals 5/6.
`True` `False`

Solution. True. Observe that 0 maps to 2, and ∞ maps to 1/3. Now 0 and ∞ lie on both (extended) coordinate axes, so the axes map to generalized circles that intersect at 2 and at 1/3. Since the transformation has real coefficients, it maps the real axis onto the real axis. Since linear fractional transformations are conformal, the imaginary axis maps to a circle passing through 2 and 1/3 that intersects the real axis orthogonally at 2. Thus the line segment joining 2 to 1/3 is a diameter of the circle. The length of the diameter is 5/3, so the radius equals 5/6.