Complex Variables

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

You may consult the textbook and your notes.

1. The final exam for this course is on what date and at what time?

   Solution. The comprehensive final exam will be held on Wednesday, May 7, at 10:30 in the morning (until 12:30 in the afternoon).

2. Evaluate the line integral $\displaystyle\int_\gamma (z + \overline{z})\,dz$, where the integration path γ is the unit circle oriented in the standard counterclockwise direction (that is, $\gamma(\theta) = e^{i\theta}$ for $0 \le \theta < 2\pi$).

   Solution. The integral equals $2\pi i$.

   Method 1. Using the parametrization gives $\int_0^{2\pi}(e^{i\theta} + e^{-i\theta})ie^{i\theta}\,d\theta = i\int_0^{2\pi}(e^{2i\theta}+1)\,d\theta = i\left[(2i)^{-1}e^{2i\theta} + \theta\right]_0^{2\pi} = 2\pi i$, since $e^{4\pi i} = e^0$.

   Method 2. On the integration path, $\overline{z}$ happens to equal $1/z$, so the integral becomes
   $$\int_{|z|=1}\left(z + \frac{1}{z}\right)\,dz = 2\pi i$$
   by Cauchy's integral formula or by the residue theorem.

3. The expression 2^i has infinitely many complex values, and all of them lie on a line. Determine complex numbers a and b such that the indicated line has the equation $\operatorname{Re}(az+b) = 0$.

   Solution. Since $2^i = e^{i(\ln 2 + 2\pi i n)} = e^{i\ln 2}e^{-2\pi n}$, where n is an integer, all of the values have argument equal to $\ln 2$. Multiplying these values by $e^{i(\frac{1}{2}\pi - \ln 2)}$ rotates them to the imaginary axis, so the equation of the line takes the form $\operatorname{Re}(ie^{-i\ln 2}z) = 0$. Thus $b = 0$, and $a = ie^{-i\ln 2}$ (or any nonzero real multiple of this value).

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4. Suppose the analytic function f maps the unit disc $\{ z : |z| < 1 \}$ into (not necessarily onto) itself. How big can $|f''(0)|$ be? Why?

Solution. The value $|f''(0)|$ never exceeds 2. The function z^2 attains this upper bound, which is therefore optimal.

The inequality $|f''(0)| \le 2$ is a special case of the Cauchy estimates for derivatives, which you established in a homework assignment (exercise 18 on page 133 in section 2.4). A proof for the case that we need here could go as follows. Fix a radius r strictly less than 1. Cauchy's integral formula says that

$$f(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w-z}\, dw \qquad \text{when } |z| < r.$$

Differentiating twice under the integral sign shows that

$$f''(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{2f(w)}{(w-z)^3}\, dw.$$

Set z equal to 0 and take absolute values:

$$|f''(0)| = \frac{1}{2\pi} \left| \int_{|w|=r} \frac{2f(w)}{w^3}\, dw \right|.$$

We are given that $|f(w)| < 1$ for all w, so our basic principle for estimating integrals (top of page 62 in the textbook) shows that

$$|f''(0)| \le \frac{1}{2\pi} \max_{|w|=r} \left(\frac{2|f(w)|}{|w^3|} \right) (2\pi r) \le \frac{2}{r^2}.$$

The left-hand side is independent of r, so taking the limit as r increases to 1 shows that $|f''(0)| \le 2$, as claimed.