Complex Variables

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

Reminder When z is a complex variable, the cosine function and the sine function are defined in terms of the exponential function via

$$\cos(z) = \tfrac{1}{2}(e^{iz} + e^{-iz}) \qquad \text{and} \qquad \sin(z) = \tfrac{1}{2i}(e^{iz} - e^{-iz}).$$

1. Determine the real and imaginary parts of $\cos(i^3)$.

   Solution. Substitute i^3 for z in the definition of the cosine function:

   $$\cos(i^3) = \tfrac{1}{2}(e^{i^4} + e^{-i^4}) = \tfrac{1}{2}(e^1 + e^{-1}).$$

   Thus the imaginary part of $\cos(i^3)$ is equal to 0, and the real part is equal to $\frac{1}{2}(e + e^{-1})$, which can also be written as $\cosh(1)$.

2. Show that if $f(z) = \cos(z)$, then $f'(z) = -\sin(z)$ (as you would expect from the corresponding differentiation formula for functions of a real variable).

   Solution. You could go back to the definition of the derivative as a limit, but it is simpler to apply the chain rule as in the example of $\sin(z)$ that we worked in class. Thus

   $$\begin{aligned}\frac{d}{dz}\cos(z) &= \tfrac{1}{2}\frac{d}{dz}(e^{iz} + e^{-iz}) = \tfrac{1}{2}(ie^{iz} - ie^{-iz}) = \tfrac{i}{2}(e^{iz} - e^{-iz}) \\ &= -\tfrac{1}{2i}(e^{iz} - e^{-iz}) = -\sin(z).\end{aligned}$$

3. Evaluate the line integral $\int_\gamma |z|^2\,dz$, where γ is the line segment from the point 0 to the point $1+i$.

   Solution. Parametrize the curve by $z = (1+i)t$, where t goes from 0 to 1. Then $|z|^2 = 2t^2$, and $dz = (1+i)\,dt$, so the integral becomes $\int_0^1 2t^2(1+i)\,dt$, or $2(1+i)\int_0^1 t^2\,dt$, which evaluates to $\frac{2}{3}(1+i)$.